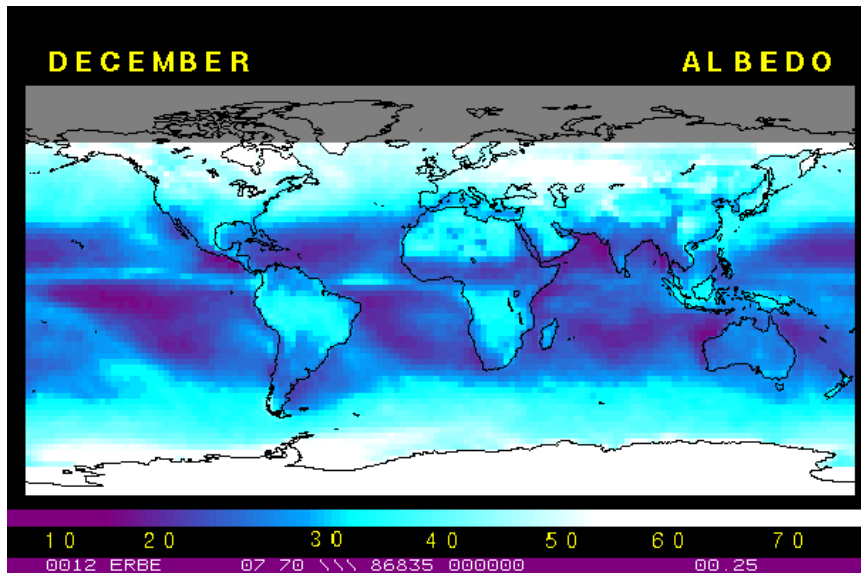


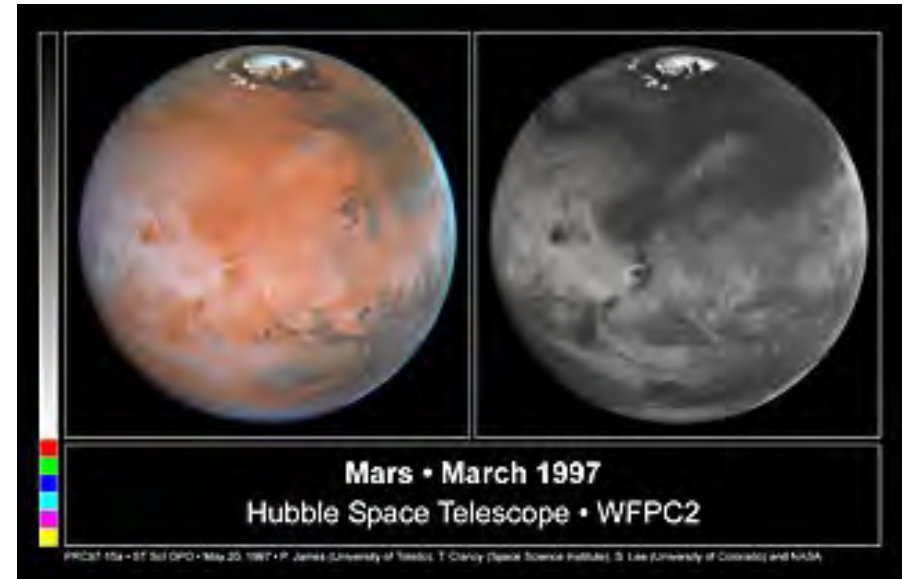
Atmospheric dynamics I

Basic equations and atmospheric waves

Clouds on planets: Origin of albedo



Earth
albedo 0.30



Mars
albedo 0.16

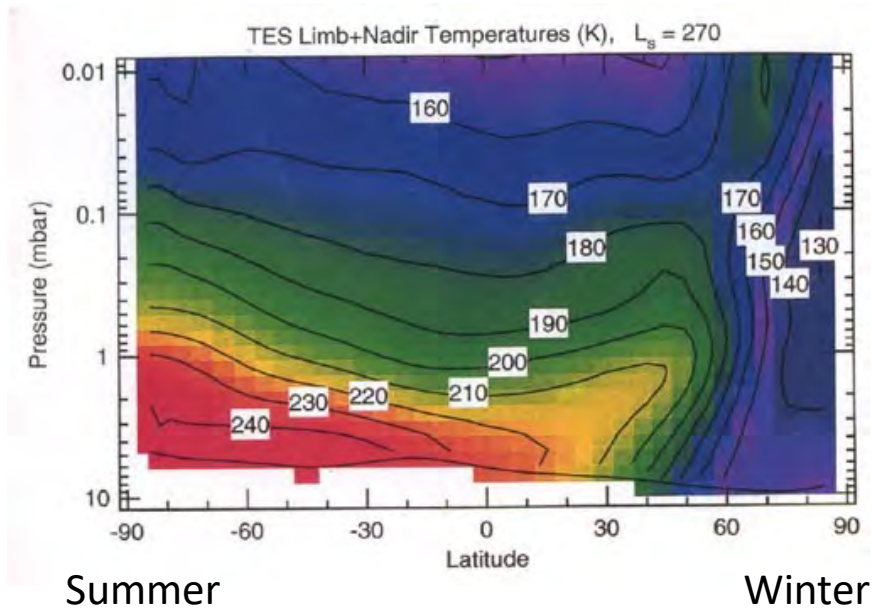


Venus
albedo 0.78

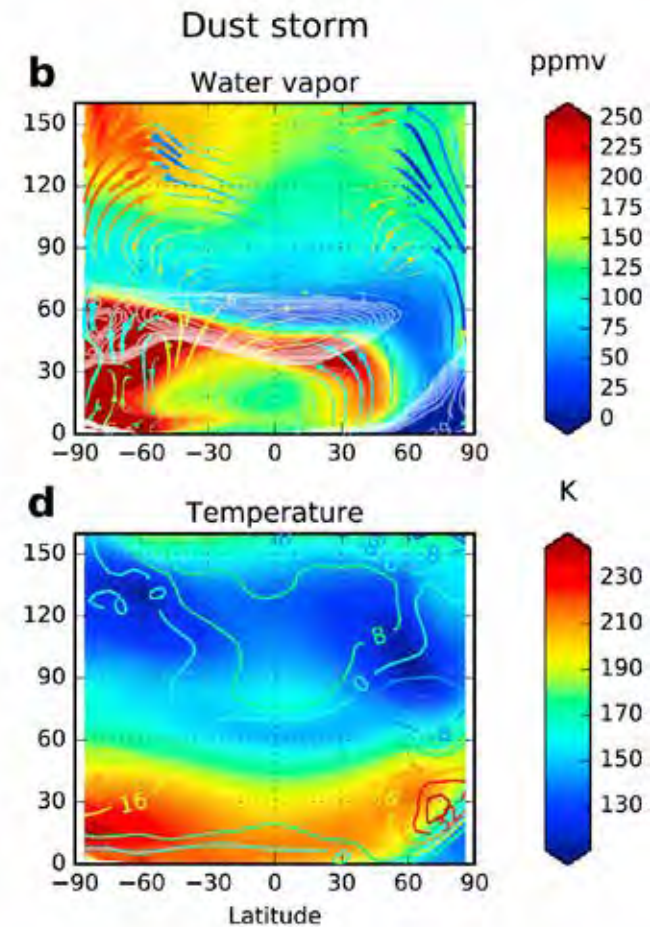
Cloud distribution reflects atmospheric dynamics

Latitudinal variation: Limit of one-dimensional model

Meridional distribution of zonal-mean temperature of Mars obtained by MGS/TES (Smith et al. 2001)

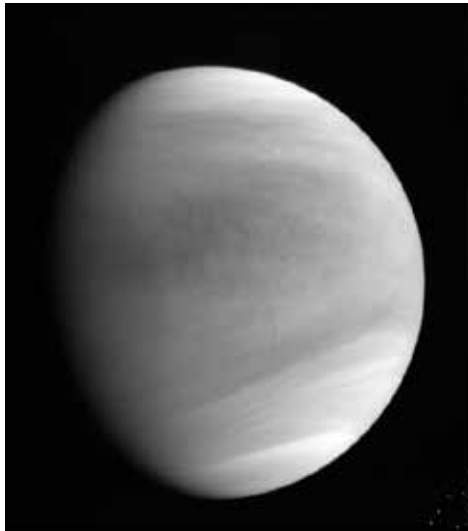


Meridional cross sections in Mars GCM (Shaposhnikov et al. 2019)



3-D circulation can play crucial roles in vertical transport

Dependence on the rotation rate?



Rotation
period 243 days



1 day



10 hours

Parameter study of the atmospheric circulation of Earth-like planets with general circulation models (GCMs)

Williams (1988)

Meridional stream function

white : anti-clockwise

shade : clockwise

Pressure

Latitude

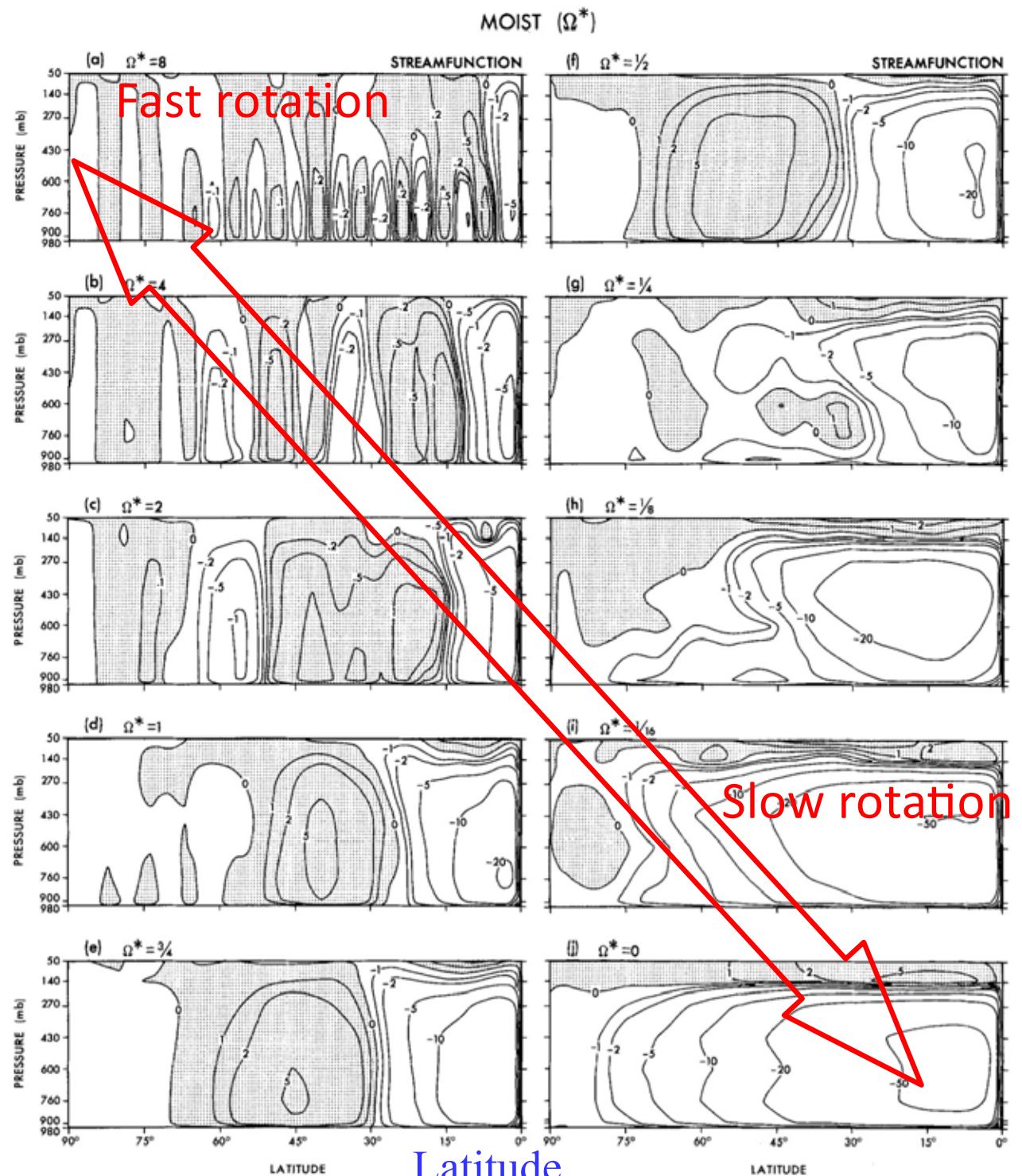


Fig. 3. Meridional distribution of the mean stream function for the MOIST model with $\Omega^* = 0-8$. Units: 10^{13} g s^{-1}

Momentum equation

Momentum equation in the inertial coordinate system

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho} \nabla p + \vec{g}$$

$\vec{v}$: velocity

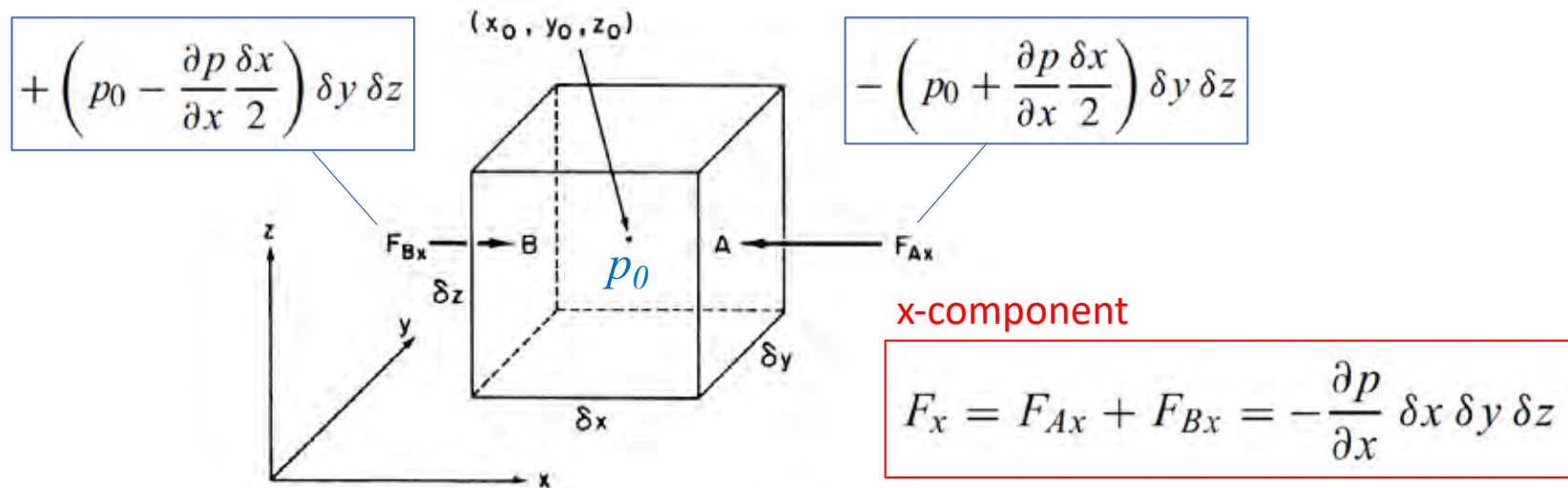
t : time

ρ : atmospheric density

p : atmospheric pressure

$\vec{g}$: gravity acceleration

Pressure gradient force



The x component of the pressure gradient force acting on a fluid element.

Holton, 2004

Momentum equation in a rotating frame

Let us consider a frame rotating at an angular velocity $\vec{\Omega}$.

For a position vector $\vec{r}$, the relationship between the time derivative in the inertial frame d/dt and that in a rotating frame d_r/dt is

$$\frac{d\vec{r}}{dt} = \frac{d_r\vec{r}}{dt} + \vec{\Omega} \times \vec{r}$$

$$\therefore \vec{v} = \vec{v}_r + \vec{\Omega} \times \vec{r} \quad \vec{v}_r : \text{velocity in rotating frame}$$

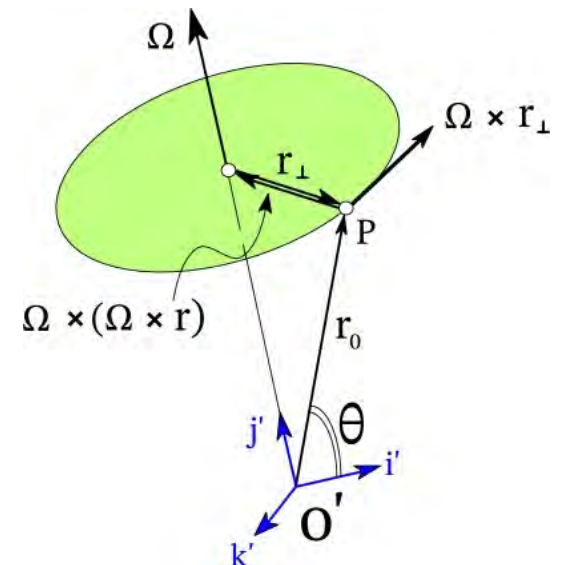
Similarly

$$\frac{d\vec{v}}{dt} = \frac{d_r\vec{v}}{dt} + \vec{\Omega} \times \vec{v}$$

Combining these, using $\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -\Omega^2 \vec{R}$ we have

$$\frac{d\vec{v}}{dt} = \frac{d_r\vec{v}_r}{dt} + 2\vec{\Omega} \times \vec{v}_r - \Omega^2 \vec{R}$$

$\vec{R}$: vector from the rotational axis to the fluid position

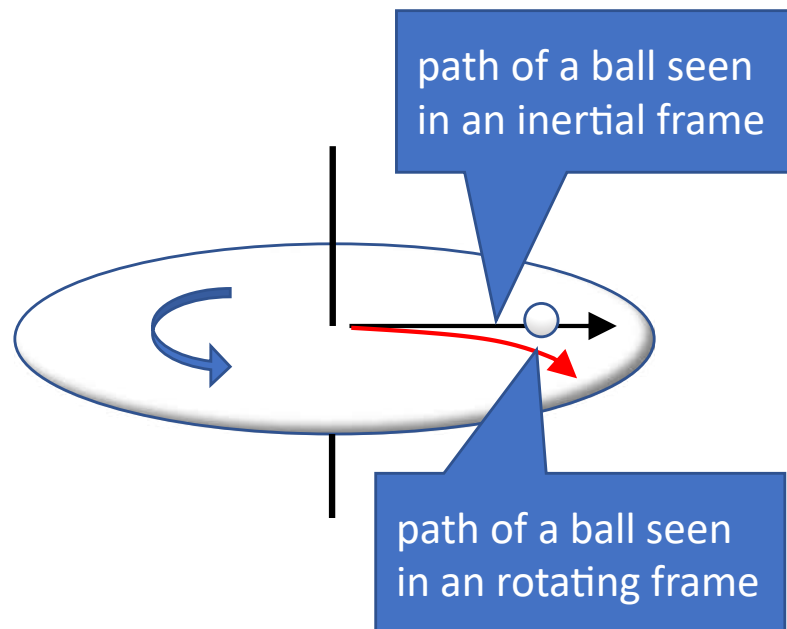


Substituting the relationship into the momentum equation, one gets

$$\frac{d_r \vec{v}_r}{dt} = -2\vec{\Omega} \times \vec{v}_r - \frac{1}{\rho} \nabla p + \vec{g} + \Omega^2 \vec{R}$$

Hereafter the subscript r is omitted. The effective gravity acceleration is defined as the sum of $\vec{g}$ and $\Omega^2 \vec{R}$. Then the momentum equation in the rotating frame is

$$\frac{\partial \vec{v}}{\partial t} = -\vec{v} \cdot \nabla \vec{v} - \underbrace{2\vec{\Omega} \times \vec{v}}_{\text{Coriolis force}} - \frac{1}{\rho} \nabla p + \vec{g}$$



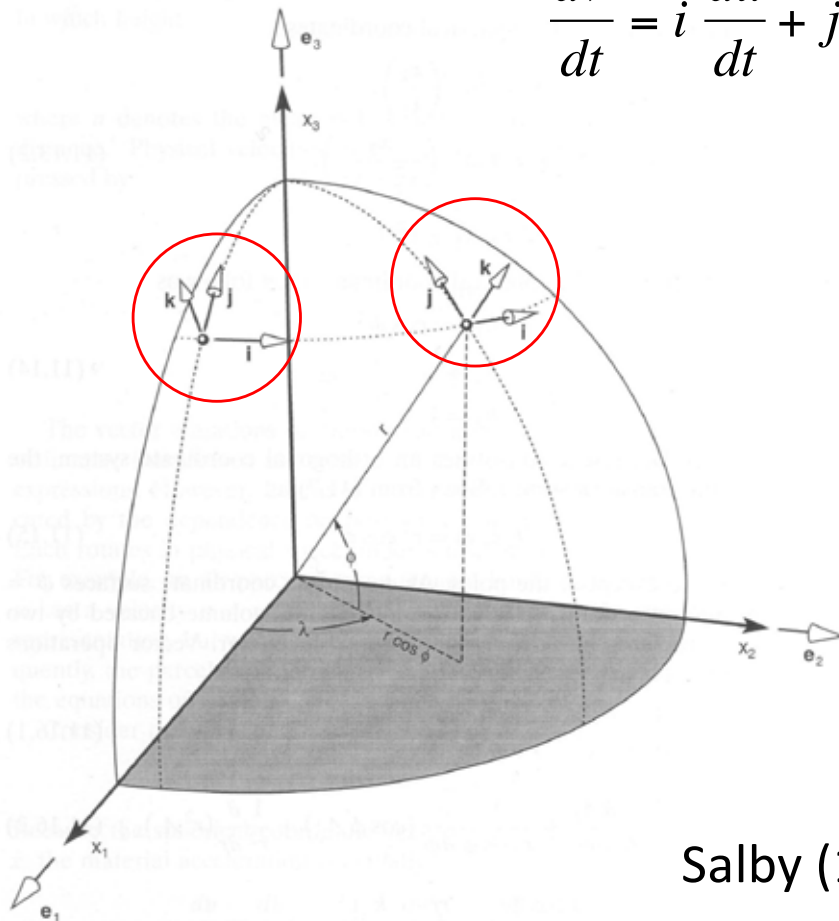
We consider an expression in a Cartesian coordinate system on a spherical surface. Using the velocity component (u, v, w) in the unit vector system $(\vec{i}, \vec{j}, \vec{k})$,

$$\vec{v} = \vec{i}u + \vec{j}v + \vec{k}w$$

$\vec{i}$: eastward, $\vec{j}$: northward, $\vec{k}$: upward

Then

$$\frac{d\vec{v}}{dt} = \vec{i} \frac{du}{dt} + \vec{j} \frac{dv}{dt} + \vec{k} \frac{dw}{dt} + u \boxed{\frac{d\vec{i}}{dt}} + v \boxed{\frac{d\vec{j}}{dt}} + w \boxed{\frac{d\vec{k}}{dt}}$$



$$\frac{d\vec{i}}{dt} = u \left(\frac{\tan \phi}{r} \vec{j} - \frac{\vec{k}}{r} \right)$$

$$\frac{d\vec{j}}{dt} = -u \frac{\tan \phi}{r} \vec{i} - \frac{v}{r} \vec{k}$$

$$\frac{d\vec{k}}{dt} = \frac{u}{r} \vec{i} + \frac{v}{r} \vec{j}$$

Salby (1996)

Figure 11.2 Spherical coordinates: longitude λ , latitude ϕ , and radial distance r . Coordinate vectors $e_\lambda = i$, $e_\phi = j$, and $e_r = k$ change with position (e.g., relative to fixed coordinate vectors e_1 , e_2 , and e_3 of rectangular Cartesian coordinates).

Momentum equations (a : planetary radius)

$$\frac{du}{dt} = \left(2\Omega \sin \phi + \frac{u \tan \phi}{a} \right) v - \frac{uw}{a} - 2\Omega w \cos \phi - \frac{1}{\rho} \frac{\partial p}{\partial x}$$

$$\frac{dv}{dt} = - \left(2\Omega \sin \phi + \frac{u \tan \phi}{a} \right) u - \frac{vw}{a} - \frac{1}{\rho} \frac{\partial p}{\partial y}$$

$$\frac{dw}{dt} = \frac{u^2 + v^2}{a} - 2\Omega u \cos \phi - \frac{1}{\rho} \frac{\partial p}{\partial z} - g$$

$$f = 2\Omega \sin \phi : \text{Coriolis parameter}$$

The deleted terms are “metric” terms arising from the spherical geometry and are negligible in many cases on the Earth.

* This is not the case for some planets such as Venus.

When $dw/dt = 0$, the equation in the vertical direction reduces to the hydrostatic equilibrium:

$$\frac{\partial p}{\partial z} = -\rho g$$

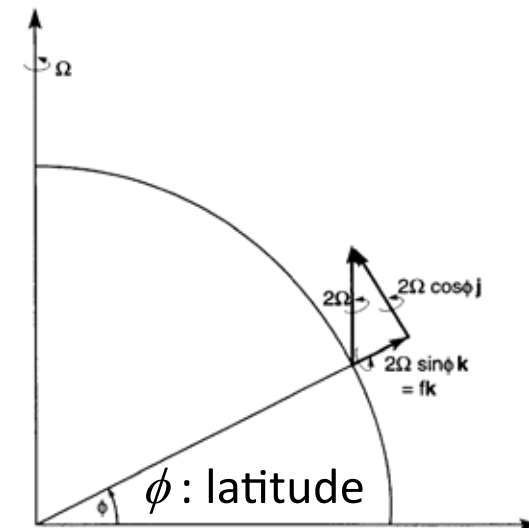


Figure 11.3 Planetary vorticity 2Ω decomposed into horizontal and vertical components.

Neglecting the metric terms and defining $\vec{v} = (u, v)$ as the horizontal component of the velocity, we have a set of governing equations:

$$\frac{\partial \vec{v}}{\partial t} = -\vec{v} \cdot \nabla \vec{v} - w \frac{\partial \vec{v}}{\partial z} - f \vec{k} \times \vec{v} - \frac{1}{\rho} \nabla p$$

horizontal momentum eq.

$$\frac{\partial w}{\partial t} = -\vec{v} \cdot \nabla w - w \frac{\partial w}{\partial z} - \frac{1}{\rho} \frac{\partial p}{\partial z} - g$$

vertical momentum eq.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{v})$$

continuity eq.

$$p = \rho R T$$

state eq.

$$\frac{\partial \theta}{\partial t} = -\vec{v} \cdot \nabla \theta - w \frac{\partial \theta}{\partial z} + \underbrace{\frac{1}{c_p} \left(\frac{p_s}{p} \right)^{R/C_p} \dot{H}}_{\text{diabatic heating}}$$

thermodynamics eq.

$\vec{k}$: unit vector in the vertical direction

$f = 2\Omega \sin \phi$: Coriolis parameter

$\dot{H}$: diabatic heating rate (J/m³/s)

Pressure coordinate

Large-scale atmospheric motions satisfy hydrostatic equilibrium.
In this case the pressure p can be used as the vertical coordinate:

$$p = p(z) \rightarrow z = z(p) \quad p : \text{pressure}, z : \text{altitude}$$

$\omega \equiv dp/dt$: vertical velocity in pressure coordinate

Φ : geopotential $d\Phi = g dz$

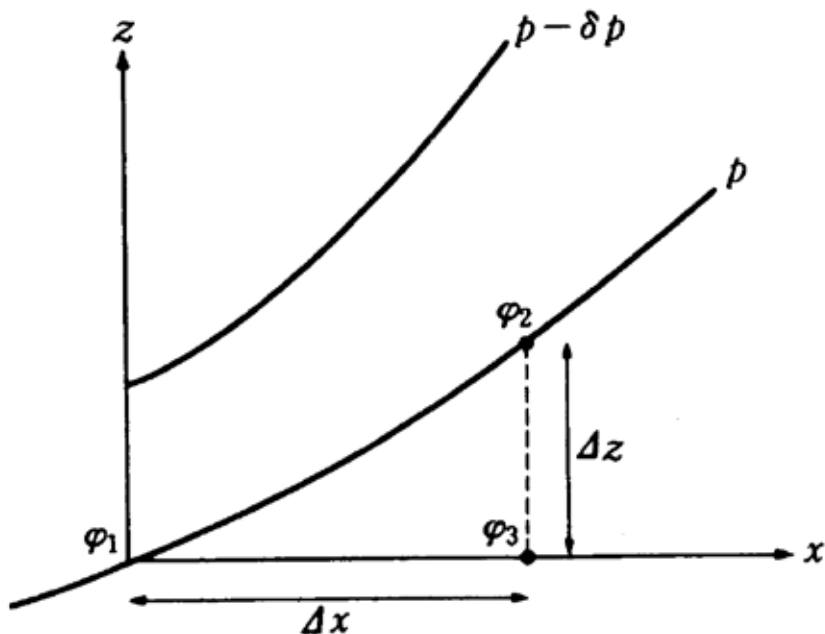


図 3.2 x - z 面内の断面. 小倉(1978)

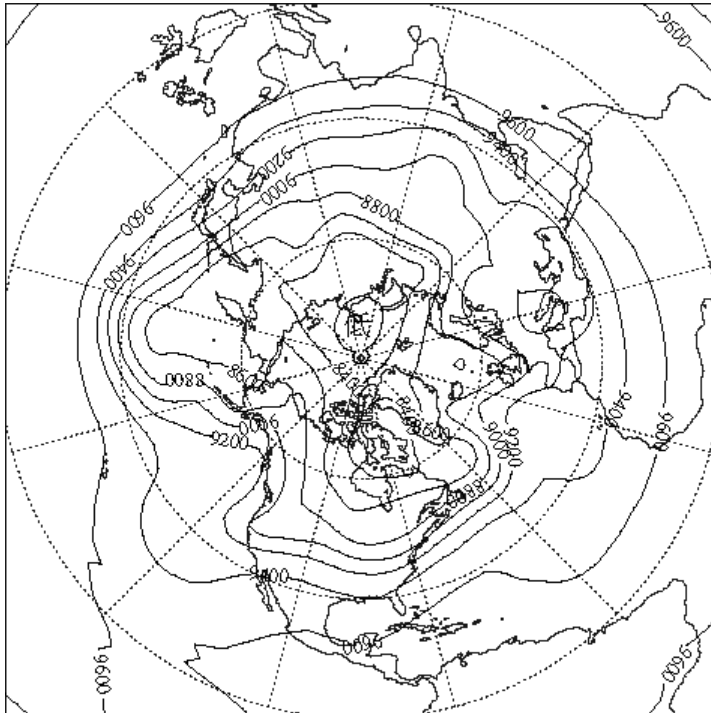
Approximate relationship
between ω and w :

$$\omega \equiv dp/dt \\ \sim -w/\rho g$$

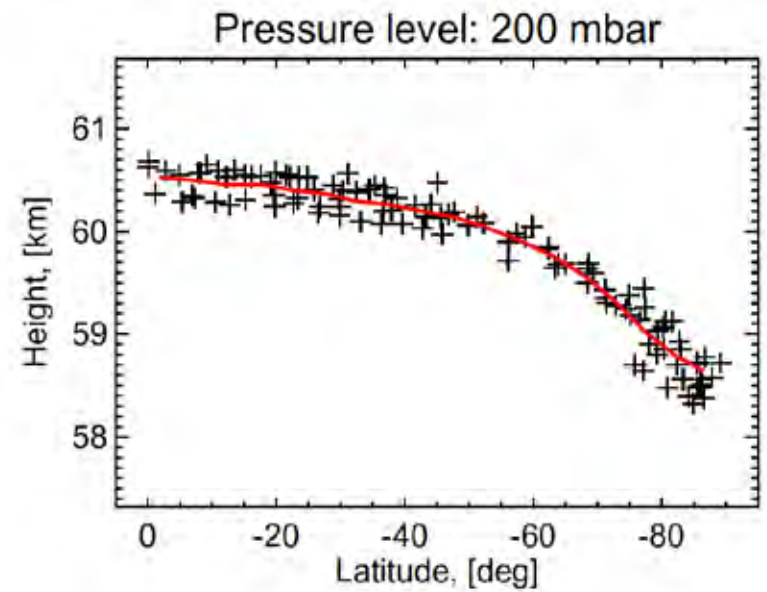
Pressure gradient force in the
momentum equation:

$$-\frac{1}{\rho} \nabla p \rightarrow -\nabla \Phi$$

Earth
300hPa height map
(2002/1/15)



Venus
Latitudinal variation of 200hPa height
(Venus Express radio occultation, by
Tellmann et al. 2009)



"Primitive equations" in pressure coordinate:

$$\frac{\partial \vec{v}}{\partial t} = -\vec{v} \cdot \nabla \vec{v} - \omega \frac{\partial \vec{v}}{\partial p} - f \vec{k} \times \vec{v} - \nabla \Phi$$

horizontal momentum eq.

$$\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}$$

state eq. + hydrostatic equilibrium

$$\nabla \cdot \vec{v} + \frac{\partial \omega}{\partial p} = 0$$

continuity eq.

$$\frac{\partial \theta}{\partial t} = -\vec{v} \cdot \nabla \theta - \omega \frac{\partial \theta}{\partial p} + \frac{1}{c_p} \left(\frac{p_s}{p} \right)^{R/C_p} \dot{H}$$

thermodynamics eq.

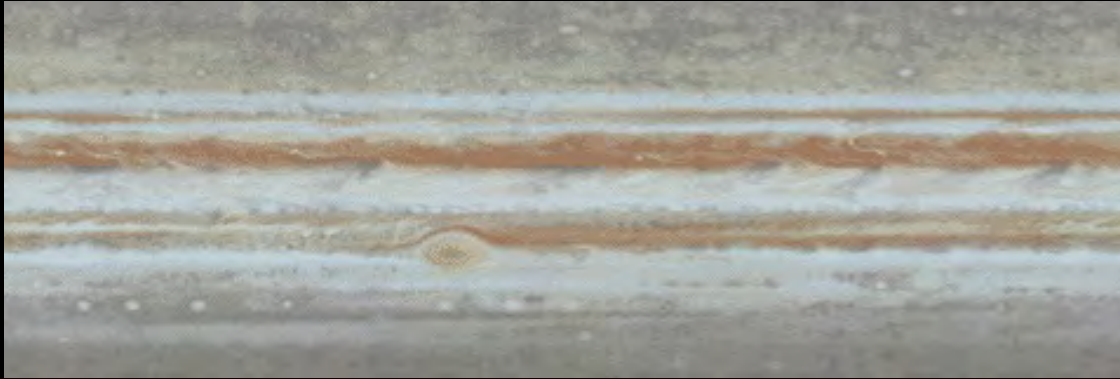
$\vec{v} = (u, v)$: horizontal velocity

ω : vertical velocity

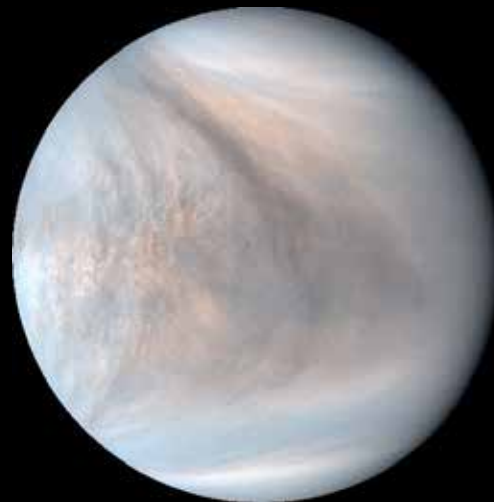
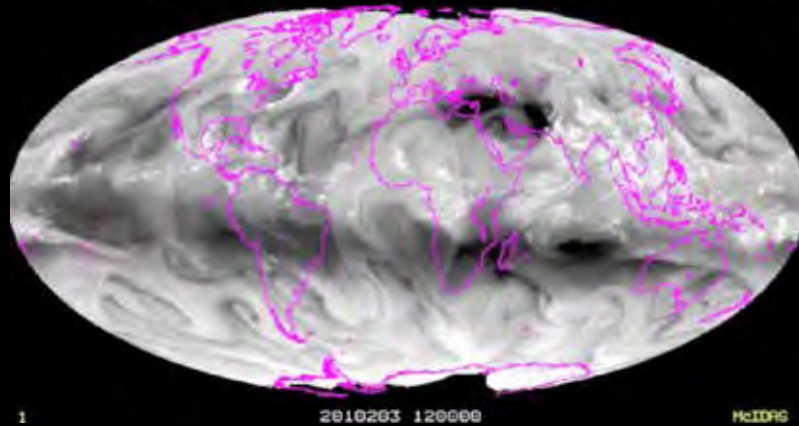
$\vec{k}$: unit vector in the vertical direction

$f = 2\Omega \sin \phi$: Coriolis parameter

$\dot{H}$: diabatic heating rate (J/m³/s)



WATERVAPOR COMPOSITE FROM 22 JUL 18 AT 12:00 UTC (SSFC/UM-MADISON)



Why do zonal currents predominate ?

How does the planetary rotation affect the dynamics?

Geostrophic flow

For synoptic scale (>1000 km) motions in Earth's atmosphere:

horizontal scale $L \sim 1000$ km

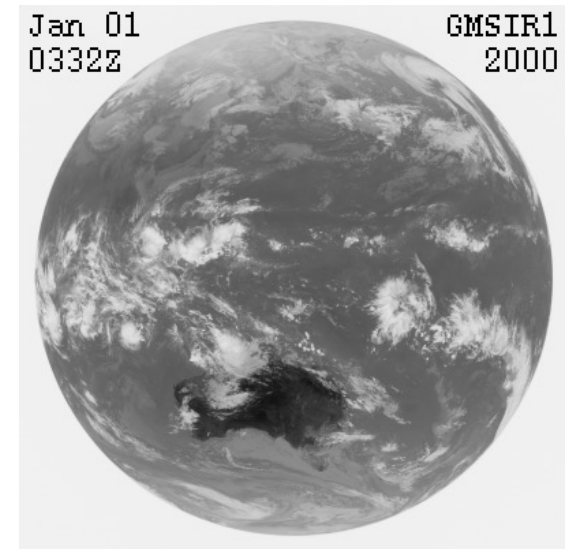
pressure scale $P \sim 1000$ hPa

velocity scale $U \sim 10$ m/s

time scale $L/U \sim 10^5$ s

Coriolis parameter $f \sim 10^{-4}$ Hz

$$\begin{array}{ccccccccc}
 \cancel{\frac{\partial \vec{v}}{\partial t}} & = & \cancel{-\vec{v} \cdot \nabla \vec{v}} & - & \cancel{\omega \frac{\partial \vec{v}}{\partial p}} & - & \underbrace{f \bar{k} \times \vec{v}}_{\text{Coriolis force}} & - & \underbrace{\nabla \Phi}_{\text{pressure gradient force}} \\
 10^{-4} & & 10^{-4} & & 10^{-4} & & 10^{-3} & & 10^{-3} \quad (\text{m s}^{-2})
 \end{array}$$



The ratio between the acceleration (U^2/L) and the Coriolis force (fU)

$$R_o \equiv \frac{U}{fL} \quad : \text{Rossby number (around 0.1 on Earth, 10–100 on Venus)}$$

Geostrophic flow is a good approximation for small Rossby numbers

$$f\bar{k} \times \vec{v}_g = -\nabla\Phi$$

Definition of geostrophic flow

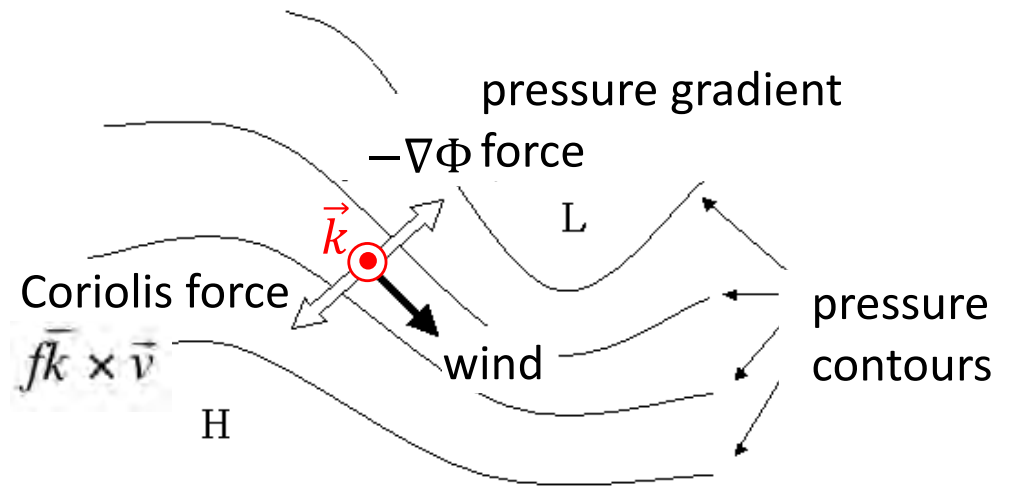
$$\vec{v}_g = \frac{1}{f} \bar{k} \times \nabla\Phi \quad \text{or} \quad u_g = -\frac{1}{f} \frac{\partial\Phi}{\partial y}, \quad v_g = \frac{1}{f} \frac{\partial\Phi}{\partial x}$$

Geostrophic flow is two-dimensional:

$$\nabla \cdot \vec{v}_g = 0$$

$$\omega = 0$$

Geostrophic flow is not a good approximation near the equator (f is small) or for scales $< O(100\text{km})$ (L is small) where $Ro (= U/fL)$ is large.



Thermal wind

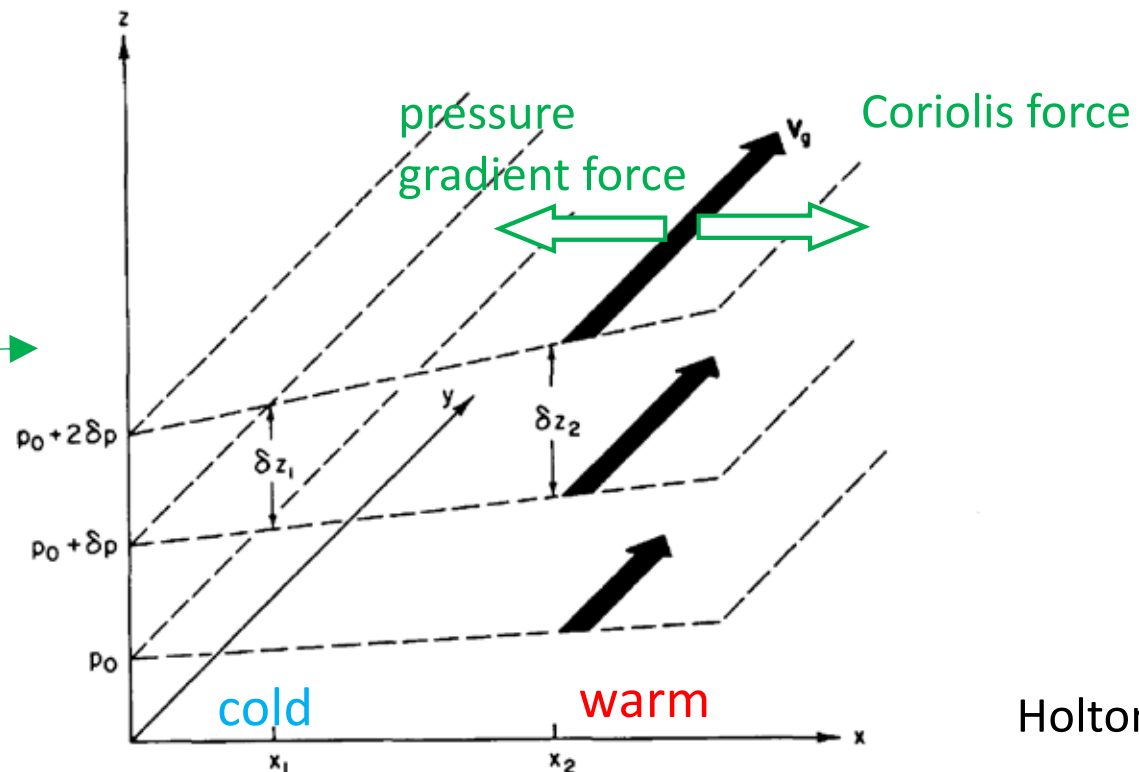
Differentiating $f\bar{k} \times \vec{v}_g = -\nabla\Phi$ with respect to p and using $\frac{\partial\Phi}{\partial p} = -\frac{RT}{p}$, “thermal wind” relation is obtained:

$$\frac{\partial \vec{v}_g}{\partial p} = -\frac{R}{f p} \bar{k} \times \nabla T$$

or

$$\frac{\partial u_g}{\partial p} = \frac{R}{f p} \frac{\partial T}{\partial y}$$

$$\frac{\partial v_g}{\partial p} = -\frac{R}{f p} \frac{\partial T}{\partial x}$$

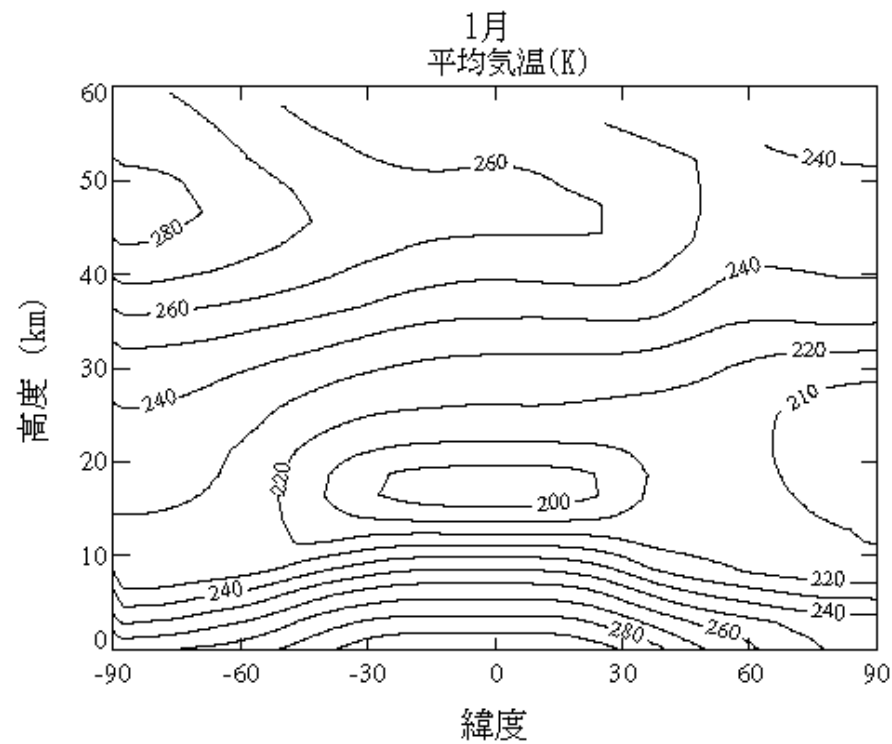


Holton (2004)

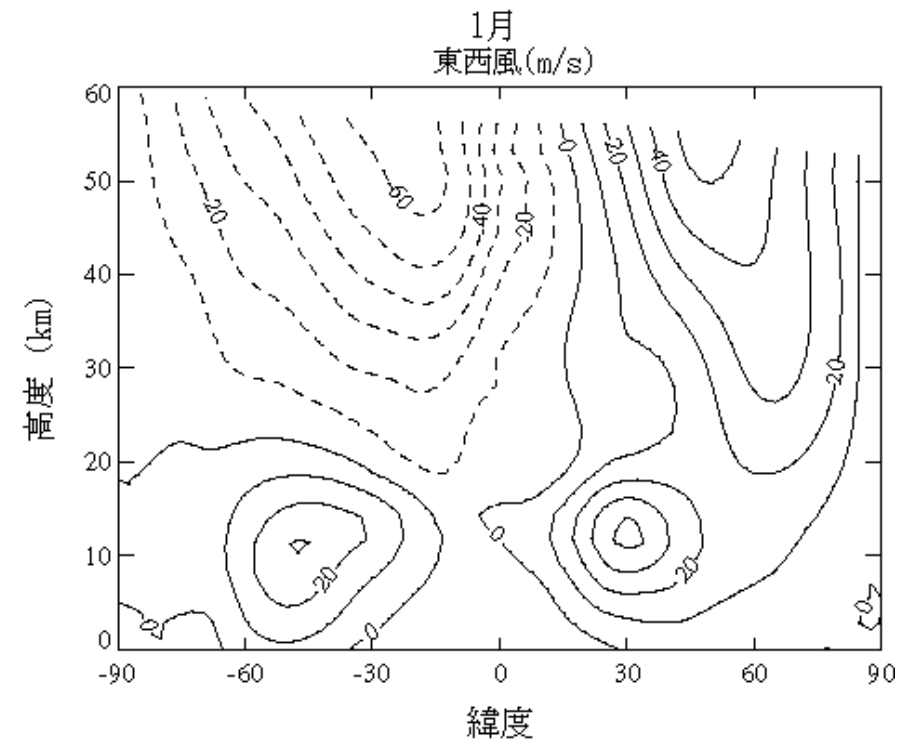
Fig. 3.8 Relationship between vertical shear of the geostrophic wind and horizontal temperature gradients. (Note: $\delta p < 0$.)

Latitude-altitude cross section of Earth's atmosphere

Temperature

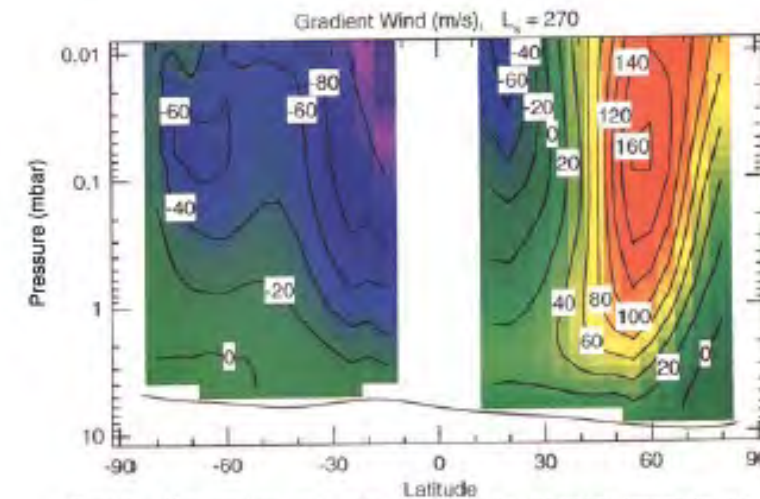
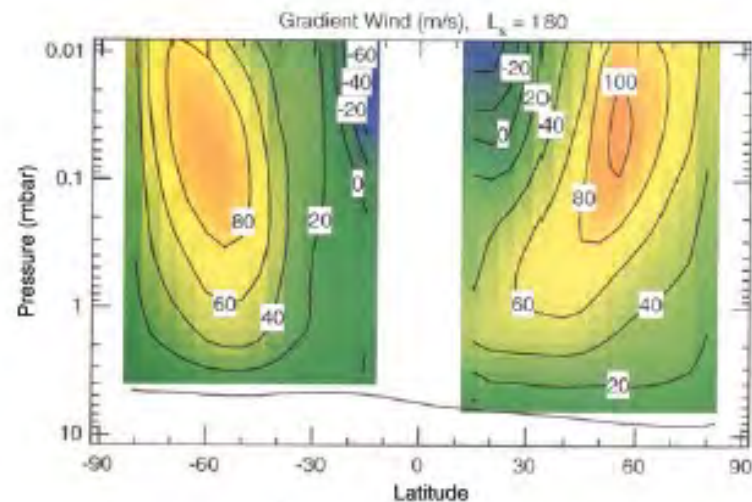
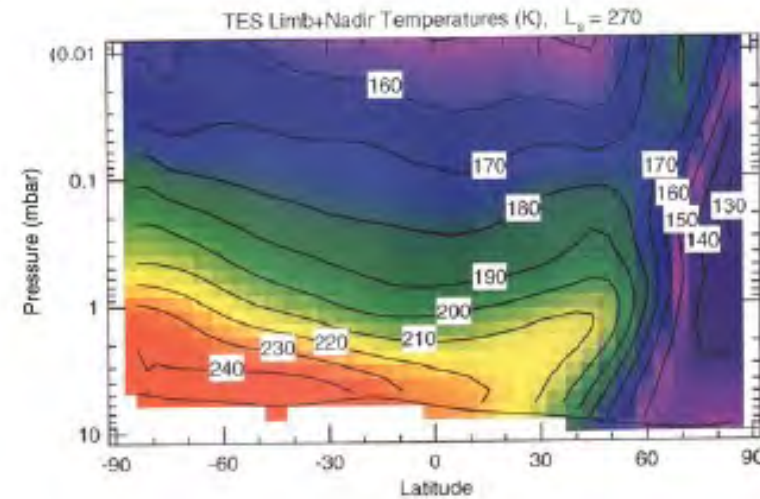
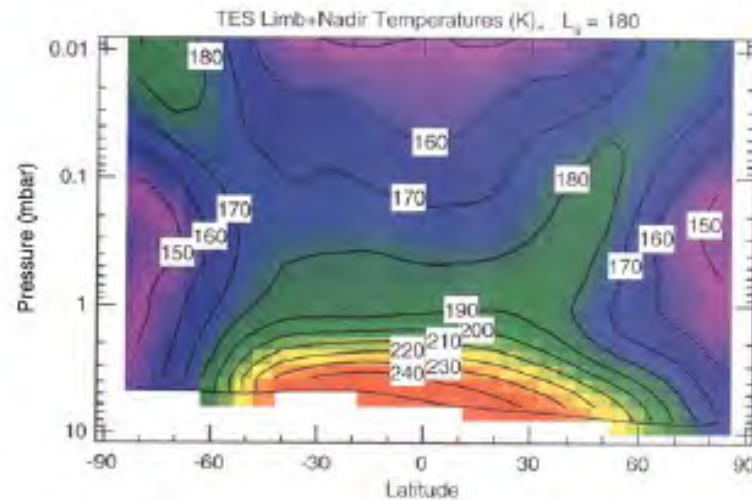


Eastward wind



Latitude-altitude cross section of Mars' atmosphere

Temperature



Eastward wind

Plate 6. Mean meridional cross sections of (top) retrieved atmospheric temperatures and (bottom) gradient wind speeds for $L_s = 180^\circ$ (northern hemisphere fall equinox). Temperatures are in kelvins, and wind speeds are in m s^{-1} . Positive wind speeds are eastward. The mean surface pressures for this data set are indicated at the lower boundary of the contoured regimes.

Plate 7. Mean meridional cross sections of (top) retrieved atmospheric temperatures and (bottom) gradient wind speeds for $L_s = 270^\circ$ (northern winter solstice). Temperatures are in kelvins, and wind speeds are in m s^{-1} . Positive wind speeds are eastward. The mean surface pressures for this data set are indicated at the lower boundary of the contoured regimes.

(Smith et al. 2001)

Geostrophic flow on Mars?

For synoptic scale (>1000 km) motions in Earth's atmosphere:

horizontal scale $L \sim 1000$ km

pressure scale $P \sim 1000$ hPa

velocity scale $U \sim 10$ m/s on Earth $\rightarrow 100$ m/s on Mars (westerly jet)

time scale $L/U \sim 10^5$ s $\rightarrow 10^4$ s on Mars

Coriolis parameter $f \sim 10^{-4}$ Hz

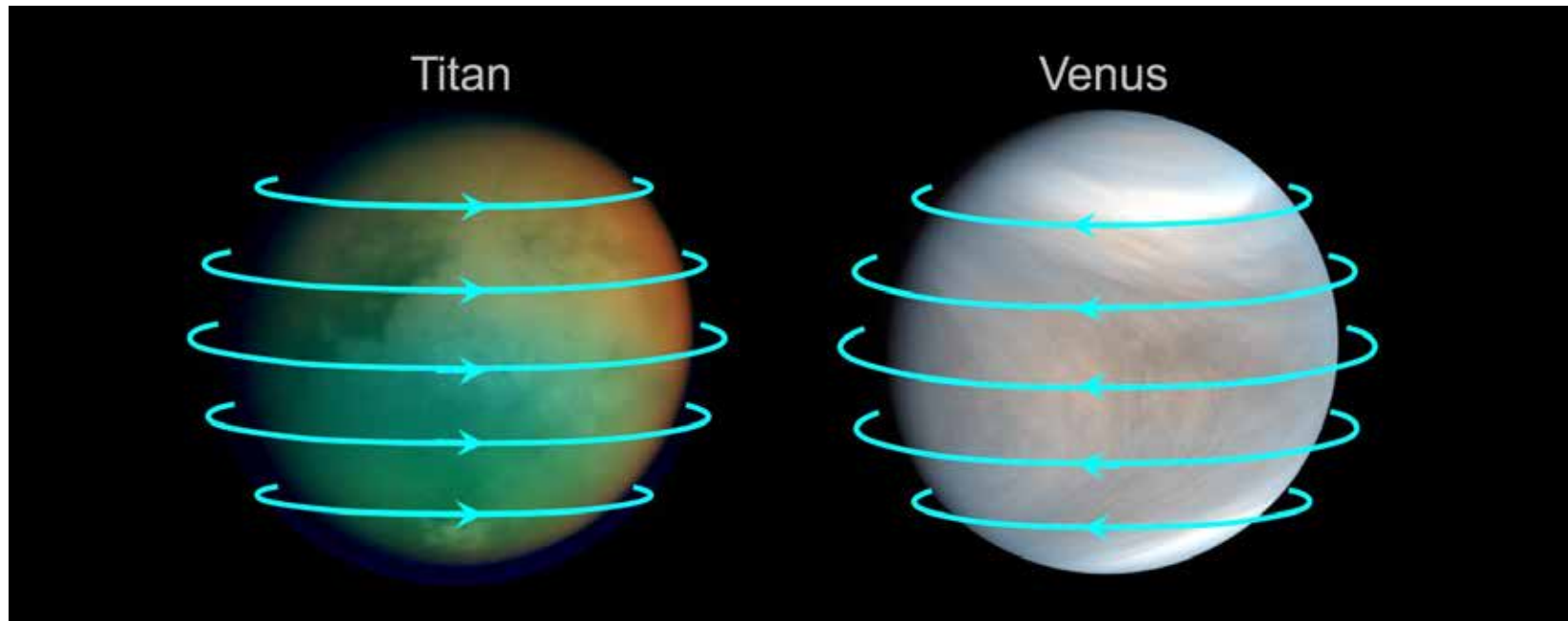
$$\begin{array}{cccccc}
 \cancel{\frac{\partial \vec{v}}{\partial t}} = \cancel{-\vec{v} \cdot \nabla \vec{v}} - \cancel{\omega \frac{\partial \vec{v}}{\partial p}} - \underbrace{f \bar{k} \times \vec{v}}_{\text{Coriolis force}} - \underbrace{\nabla \Phi}_{\text{pressure gradient force}} \\
 \\
 \begin{array}{ccccc}
 10^{-4} & 10^{-4} & 10^{-4} & 10^{-3} & 10^{-3} & (\text{m s}^{-2}) \text{ on Earth} \\
 10^{-2} & 10^{-2} & 10^{-2} & 10^{-2} & 10^{-2} & (\text{m s}^{-2}) \text{ on Mars}
 \end{array}
 \end{array}$$

The ratio between the acceleration (U^2/L) and the Coriolis force (fU)

$$R_o \equiv \frac{U}{fL} \quad : \text{ Rossby number (around 0.1 on Earth, 10–100 on Venus)}$$

around 1 on Mars ?

Superrotation

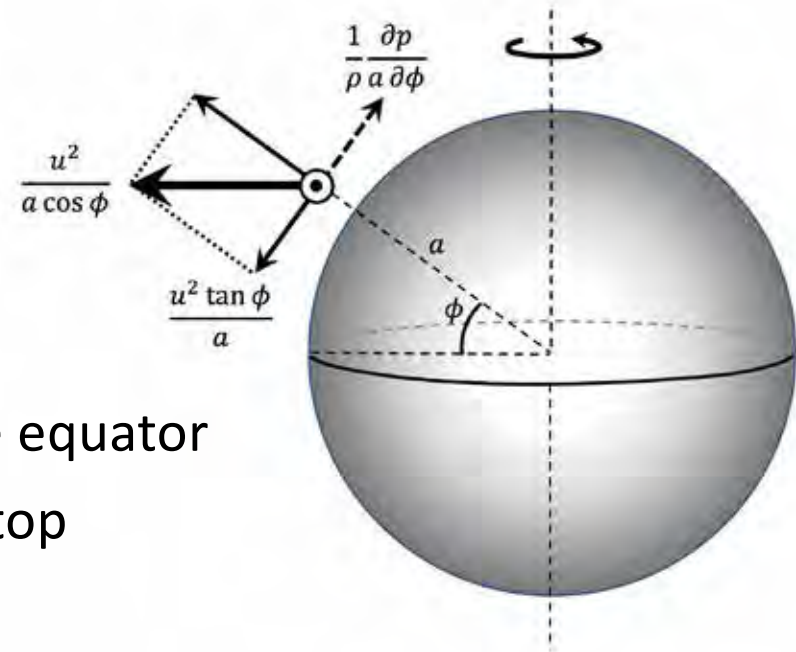


Superrotation

It prevails in Venus's atmosphere

planetary rotation: 1.8 m/s on the equator

zonal wind: 100 m/s at the cloud top



Momentum equations

$$\frac{du}{dt} = \left(\cancel{2\Omega \sin \phi} + \boxed{\frac{u \tan \phi}{a}} \right) v - \cancel{\frac{uw}{a}} - \cancel{2\Omega w \cos \phi} - \frac{1}{\rho} \frac{\partial p}{\partial x}$$

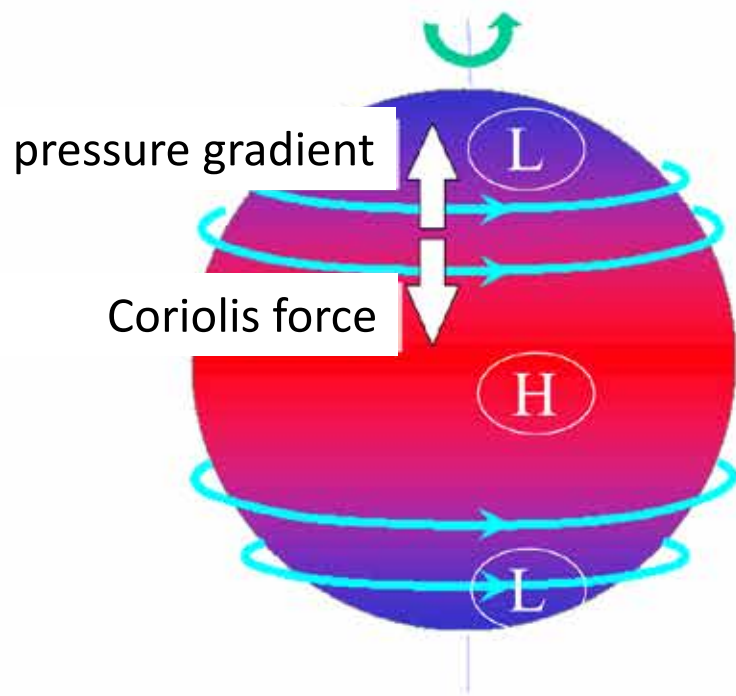
$$\frac{dv}{dt} = - \left(\cancel{2\Omega \sin \phi} + \boxed{\frac{u \tan \phi}{a}} \right) u - \cancel{\frac{vw}{a}} - \frac{1}{\rho} \frac{\partial p}{\partial y}$$

$$\frac{dw}{dt} = \cancel{\frac{u^2 + v^2}{a}} - \cancel{2\Omega u \cos \phi} - \frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad f = 2\Omega \sin \phi$$

Atmospheric rotation takes the place of planetary rotation.

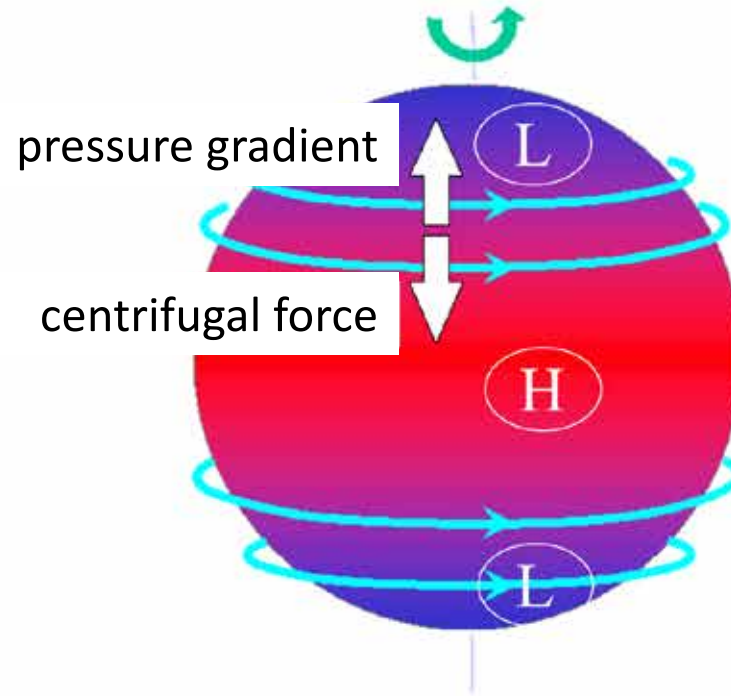
Meridional force balance of zonal flow

Geostrophic flow 地衡風
(planetary rotation $\gg$ wind)



Earth-like

Cyclostrophic flow 旋衡風
(planetary rotation $\ll$ wind)



Venus-like

Thermal wind

Geostrophic flow
(planetary rotation $\gg$ wind)

Cyclostrophic flow
(planetary rotation $\ll$ wind)

$$\cancel{\frac{dv}{dt}} = - \left(\boxed{2\Omega \sin \phi} + \boxed{\frac{u \tan \phi}{a}} \right) u - \cancel{\frac{vw}{a}} - \frac{1}{\rho} \frac{\partial p}{\partial y}$$

$$2\Omega \sin \phi a u + \frac{1}{\rho} \frac{\partial p}{\partial \phi} = 0$$

$$u^2 \tan \phi + \frac{1}{\rho} \frac{\partial p}{\partial \phi} = 0$$

← state eq.

← hydrostatic equilibrium

$$\frac{\partial u}{\partial \zeta} + \frac{R}{2a\Omega \sin \phi} \frac{\partial T}{\partial \phi} = 0$$

$$\frac{\partial u^2}{\partial \zeta} + \frac{R}{\tan \phi} \frac{\partial T}{\partial \phi} = 0$$

$\zeta = -\ln p$: log-pressure altitude

Latitude-altitude cross section of Venus's atmosphere

Temperature

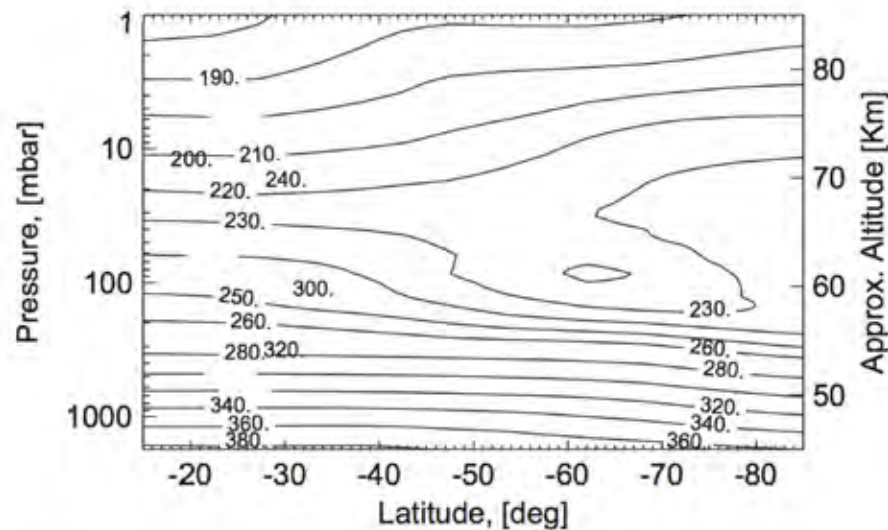


Fig. 2. Meridional cross section of atmospheric temperature (K) obtained combining 116 VeRa profiles. Hemispherical symmetry and local time independence has been assumed. Contours have been smoothed. Contours interval is 10 K, some contours have been removed to render the plot clearer.

Westward wind

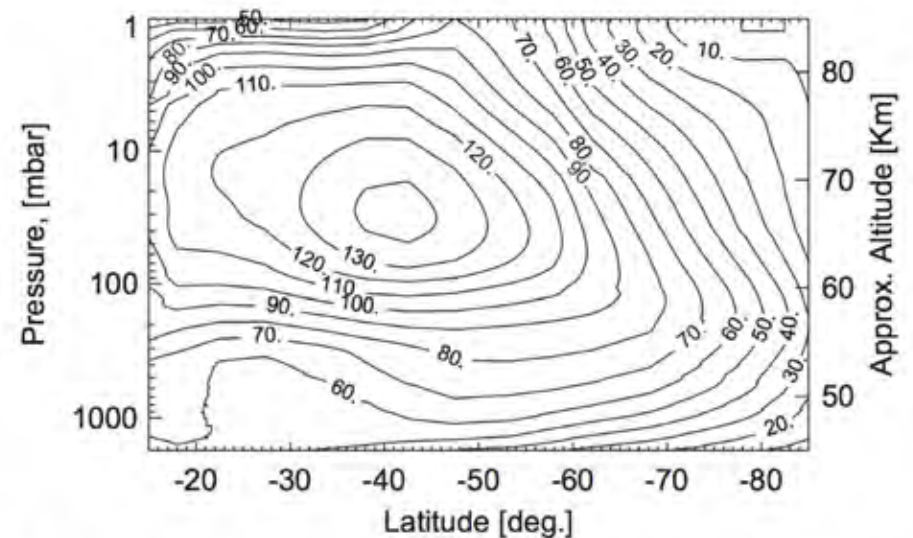
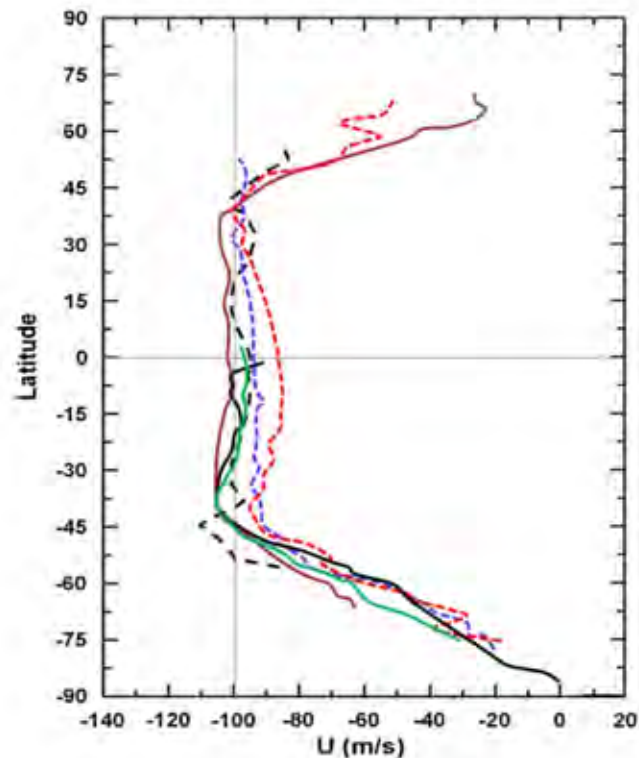


Fig. 8. Latitude-height cross section of zonal thermal wind speed (m s^{-1}) derived from VeRa temperature profiles assuming cyclostrophic balance (Eq. (5)). The velocity used as lower boundary condition is the cloud-tracked wind profile retrieved from VIRTIS/VEx 1.74 μm images (Sánchez-Lavega et al., 2008). Contour interval is 10 m s^{-1} .

(Piccialli et al., 2012)

Latitude-altitude cross section of Venus's atmosphere

Cloud-tracked zonal winds around 70 km
(Sánchez-Lavega et al. 2017)



Zonal winds determined from cyclostrophic balance (Piccialli et al., 2012)

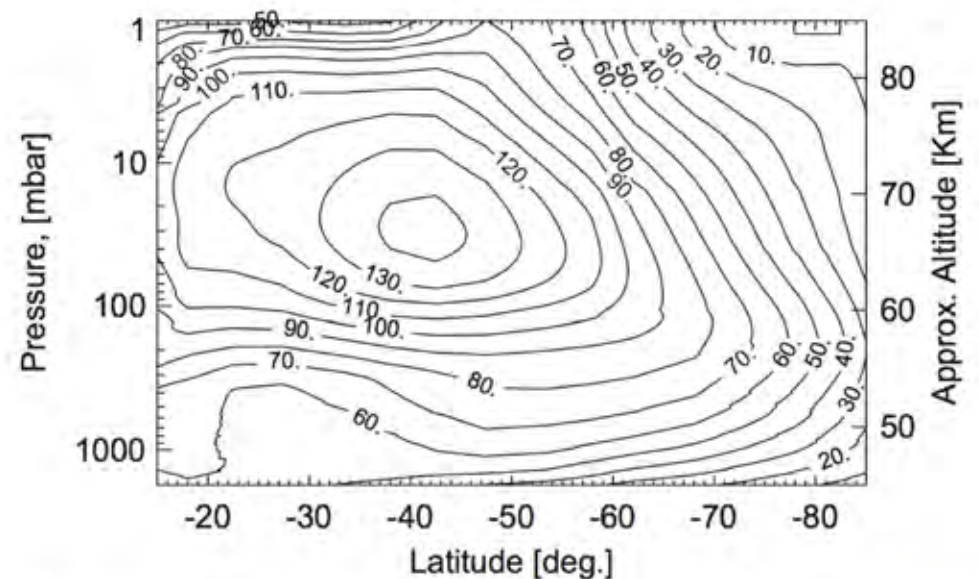


Fig. 8. Latitude–height cross section of zonal thermal wind speed (m s^{-1}) derived from VeRa temperature profiles assuming cyclostrophic balance (Eq. (5)). The velocity used as lower boundary condition is the cloud-tracked wind profile retrieved from VIRTIS/VEx $1.74 \mu\text{m}$ images (Sánchez-Lavega et al., 2008). Contour interval is 10 m s^{-1} .

The discrepancy is not understood

Radiative energy budget and meridional circulation

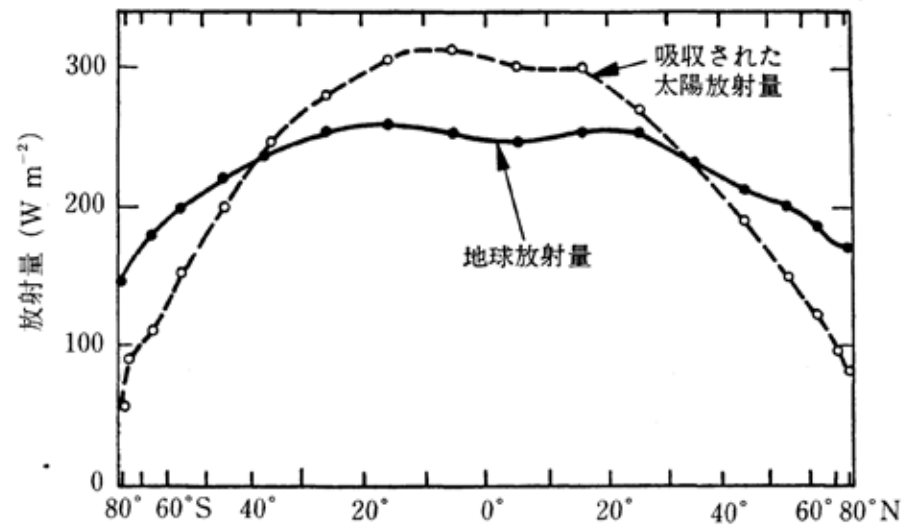


図 4.4 1962～66 年の衛星観測に基づく帯状平均した放射収支の緯度分布 (Vonder Haar and V. Suomi, 1971 : *J. Atmos. Sci.*, 28, 305-314.)

小倉(1978)

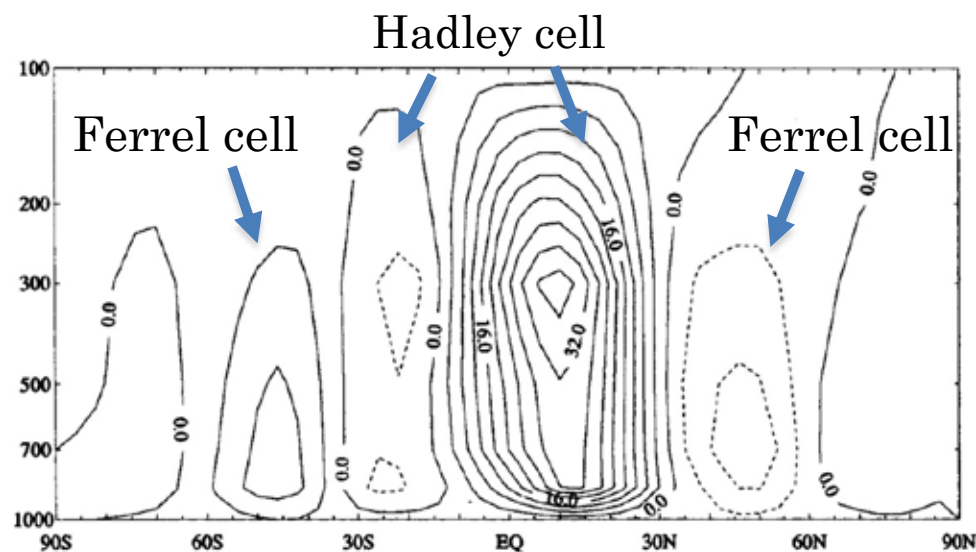


Fig. 10.7 Streamfunction (units: $10^2 \text{ kg m}^{-1} \text{ s}^{-1}$) for the observed Eulerian mean meridional circulation for Northern Hemisphere winter, based on the data of Schubert et al. (1990).

Holton (2004)

Radiative energy budget and meridional circulation

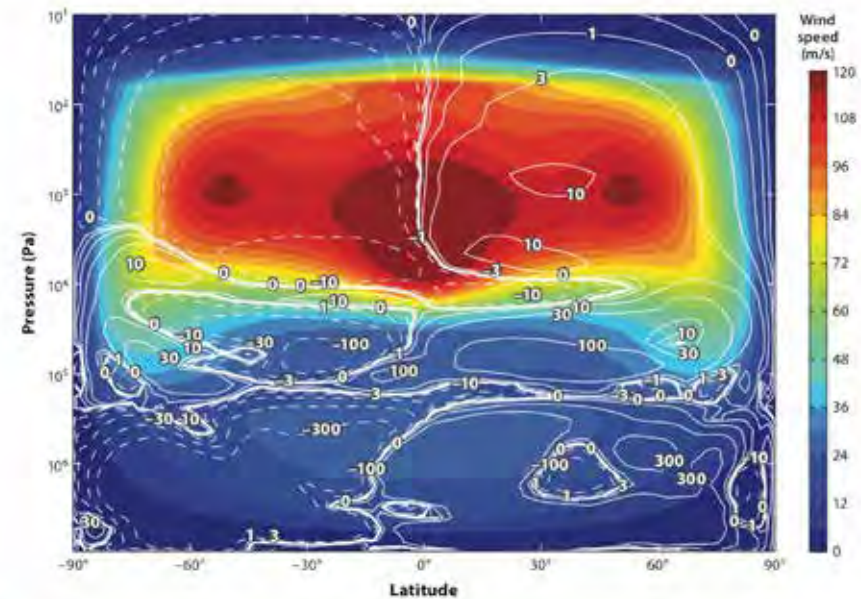
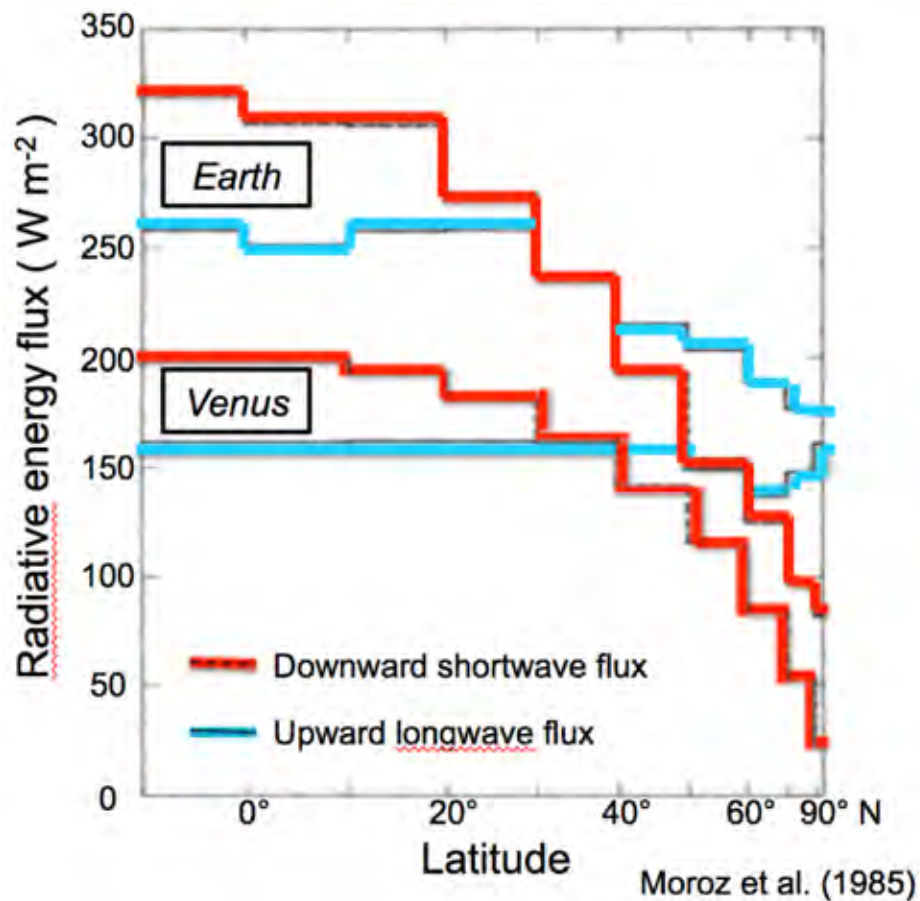
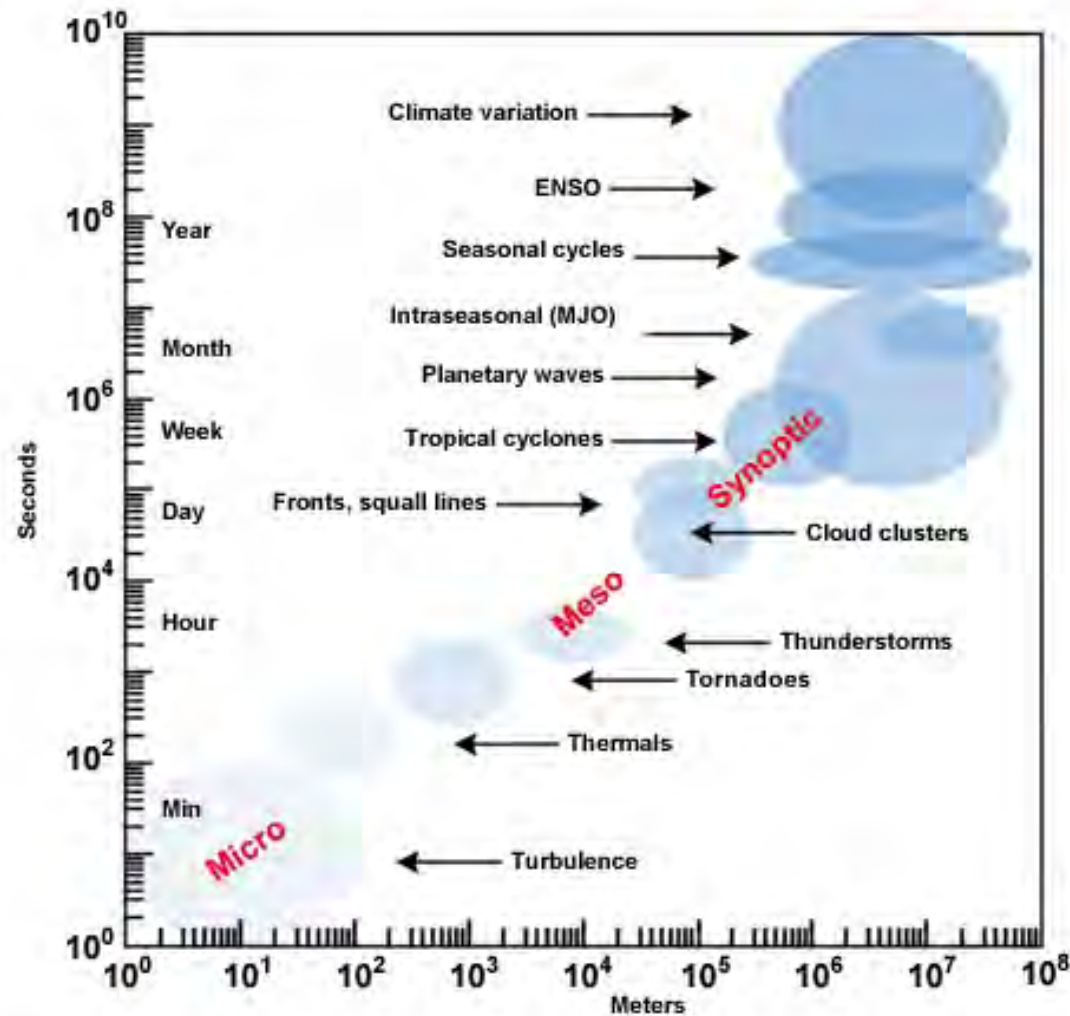


Figure 5

Venus modeled zonal-mean zonal winds, modified from Lebonnois et al. (2016) with permission from Elsevier. White contours show the averaged stream function (i.e., the mean meridional circulation) in units of 10^9 kilograms per second, solid contours indicate clockwise rotation, and dashed contours indicate counterclockwise rotation.

Read & Lebonnois
(2018)

Space and time-scales of dynamical atmospheric processes

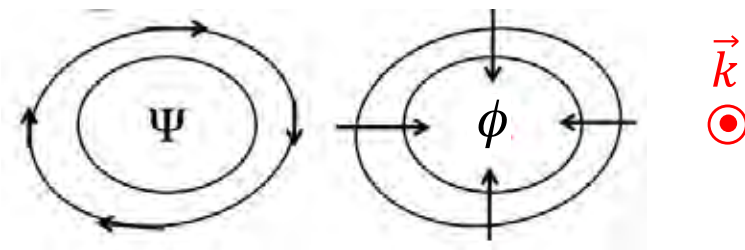


©The COMET Program

Rotational wind and divergent wind

The horizontal velocity vector can be expressed with the stream function ψ and the velocity potential ϕ :

$$\vec{v} = \underbrace{\vec{k} \times \nabla \psi}_{\text{Rotational wind}} + \underbrace{\nabla \phi}_{\text{Divergent wind}}$$



Rotational wind is nondivergent

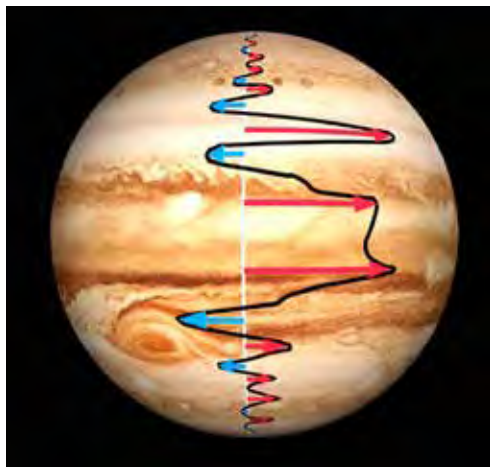
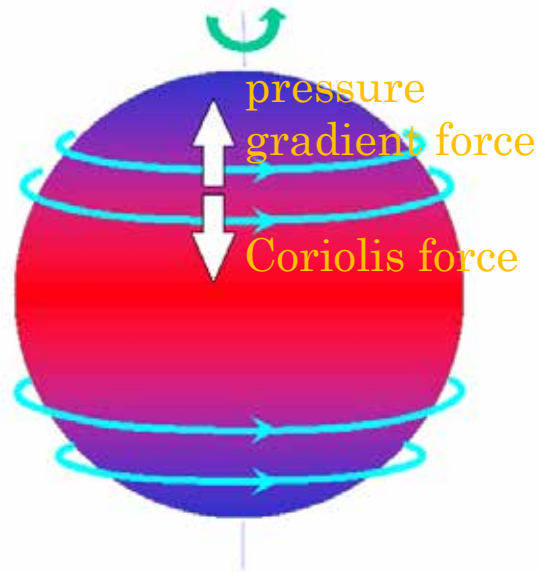
$$\nabla \cdot (\vec{k} \times \nabla \psi) = 0 \quad : \text{geostrophic flow, Rossby wave}$$

Divergent wind is irrotational

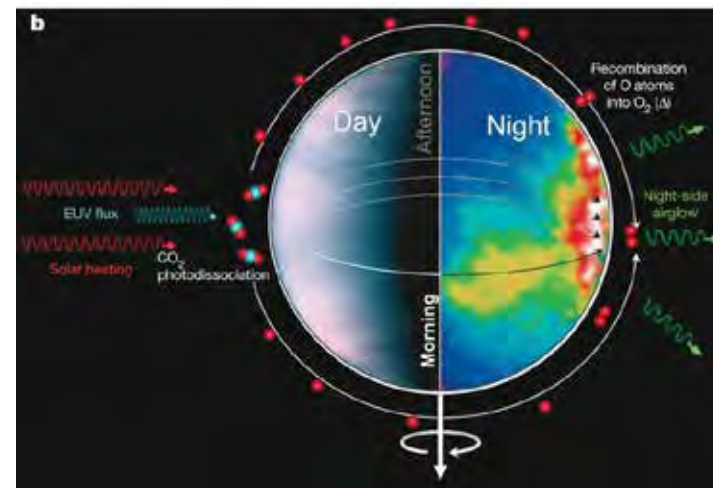
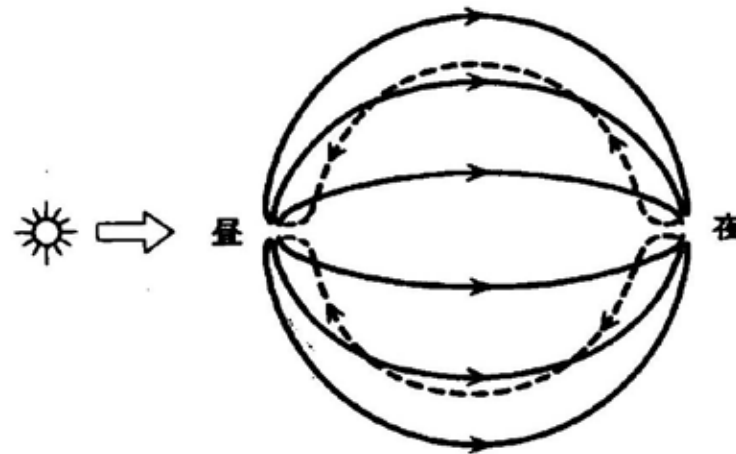
$$\nabla \times \nabla \phi = 0 \quad : \text{convection, gravity wave}$$

Planetary-scale motions

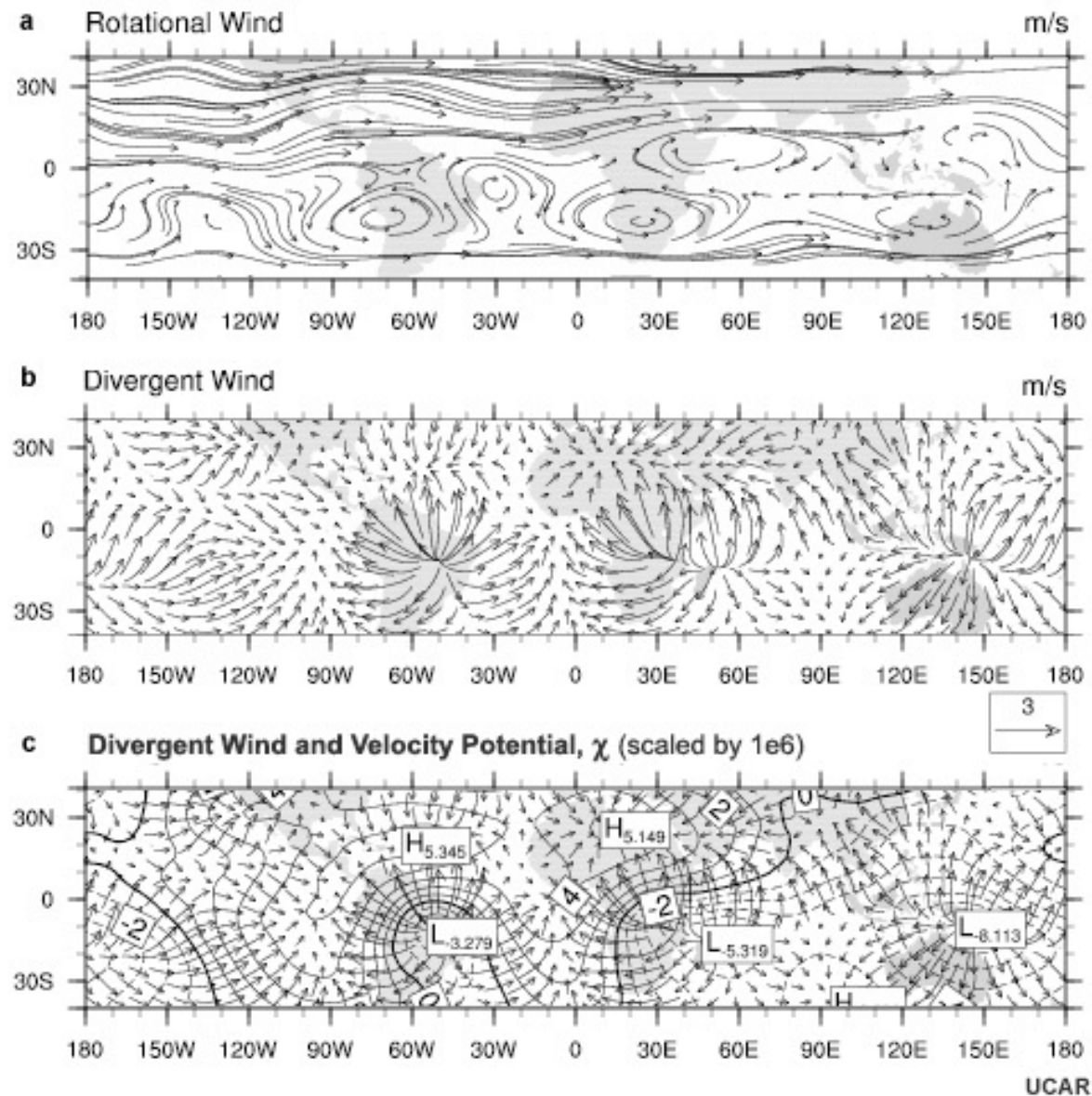
(rotational flow)
Mean zonal wind



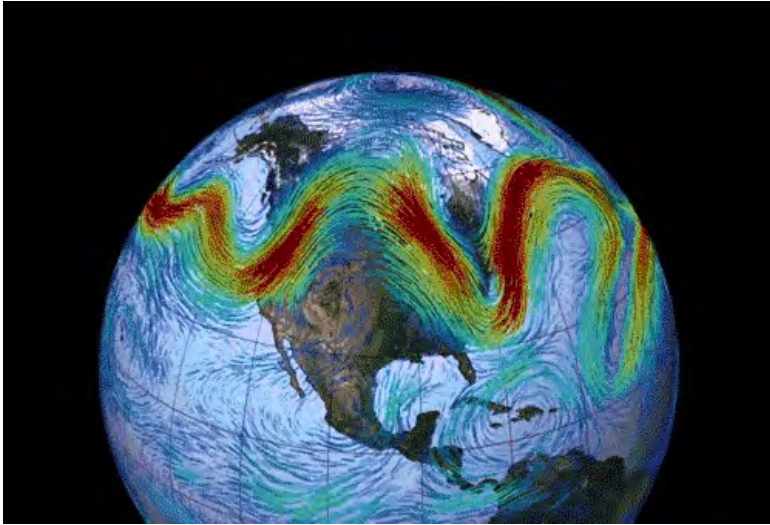
(divergent flow)
Hadley circulation
Subsolar-to-antisolar circulation



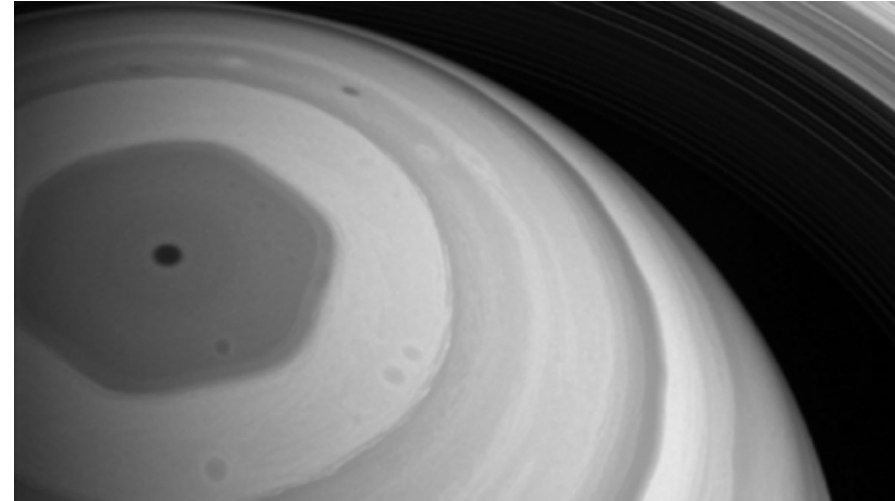
Rotational and divergent winds on Earth



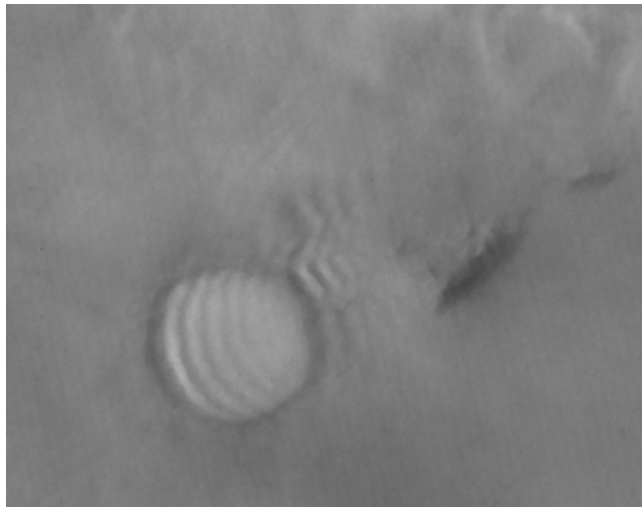
Atmospheric waves



Earth



Saturn



Mars

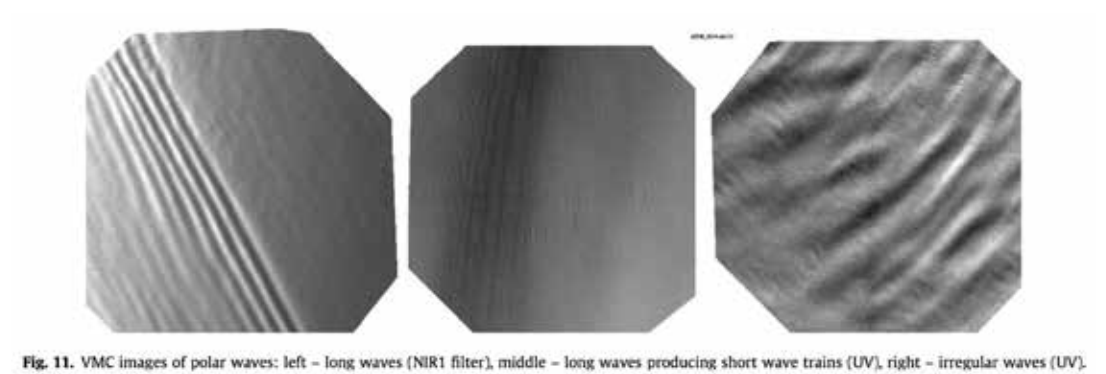


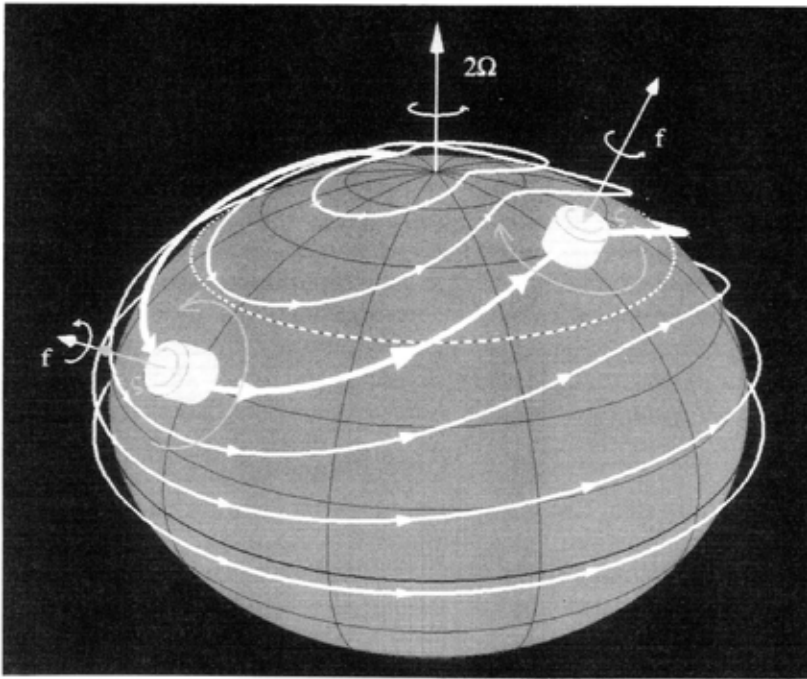
Fig. 11. VMC images of polar waves: left – long waves (NIR1 filter), middle – long waves producing short wave trains (UV), right – irregular waves (UV).

Venus

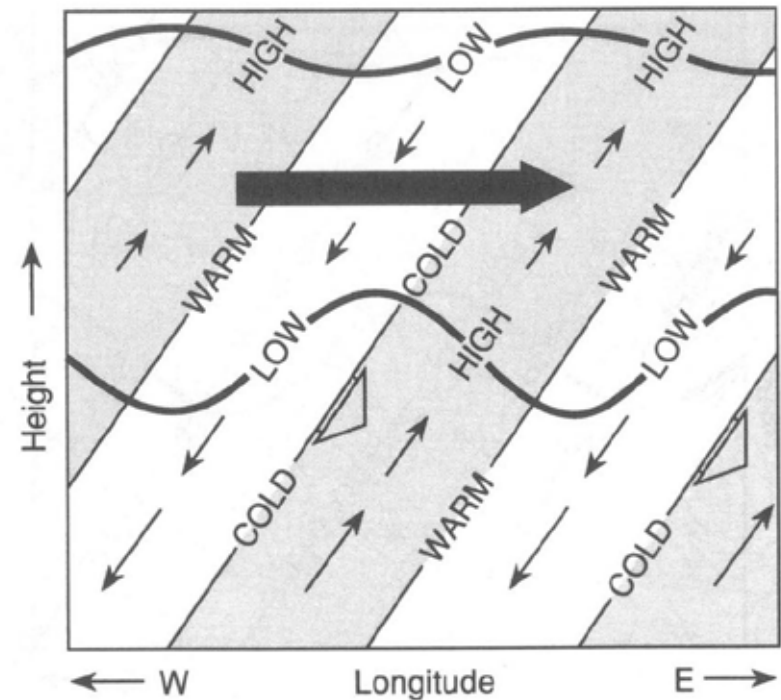
Atmospheric waves

- generated in an unstable background atmosphere
- transport momentum and energy over long distances
- induce mixing

Rossby wave (horizontal oscillation)



Gravity wave (vertical oscillation)



Salby (1996)

Vorticity

The *circulation* C about a closed contour is defined as the line integral along the contour of the component of the velocity vector that is locally tangent to the contour:

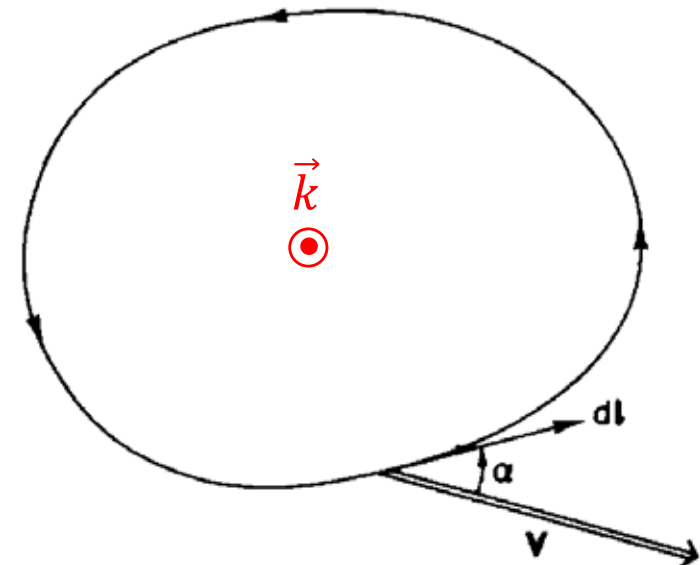
$$C = \oint_l \vec{v} \cdot d\vec{l} = \oint_l |\vec{v}| \cos \alpha dl = \iint_S (\nabla \times \vec{v}) \cdot \vec{n} dS$$

n : unit vector normal to the surface

(Stokes' theorem was used.)

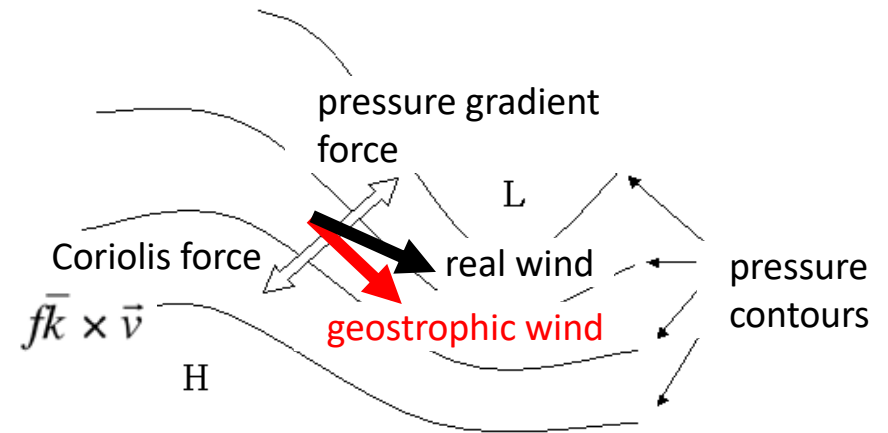
Dividing the circulation δC encircling a small area δS , and considering the limit $\delta S \rightarrow 0$, we get the *vorticity*:

$$\begin{aligned} \xi &= \frac{\delta C}{\delta S} = (\nabla \times \vec{v}) \cdot \vec{k} \\ &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{aligned}$$



Holton (1992)

Quasi-geostrophic approximation



The real wind is divided into **geostrophic** wind and **ageostrophic** wind.

$$\vec{v} = \boxed{\vec{v}_g} + \boxed{\vec{v}_a}$$

$$\vec{v}_g \equiv \frac{1}{f_0} \vec{k} \times \nabla \Phi$$

$$\frac{v_a}{v_g} = O(R_o) \ll 1$$

$$R_o \equiv \frac{U}{fL}$$

The ratio of the magnitudes of the ageostrophic and geostrophic winds is the same order of magnitude as the Rossby number (around 0.1 in Earth's atmosphere)

Lagrangian derivative:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla + \omega \frac{\partial}{\partial p}$$

The advection term is approximated by advection by geostrophic wind:

$$\frac{d_g}{dt} \equiv \frac{\partial}{\partial t} + \boxed{\vec{v}_g} \cdot \nabla$$

Beta-plane approximation

The first-order Taylor series approximation of the Coriolis parameter f :

$$\begin{aligned} f &= 2\Omega \sin \phi \\ &\sim 2\Omega [\sin \phi_0 + (\phi - \phi_0) \cos \phi_0] \\ &= \boxed{f_0 + \beta y} \end{aligned}$$

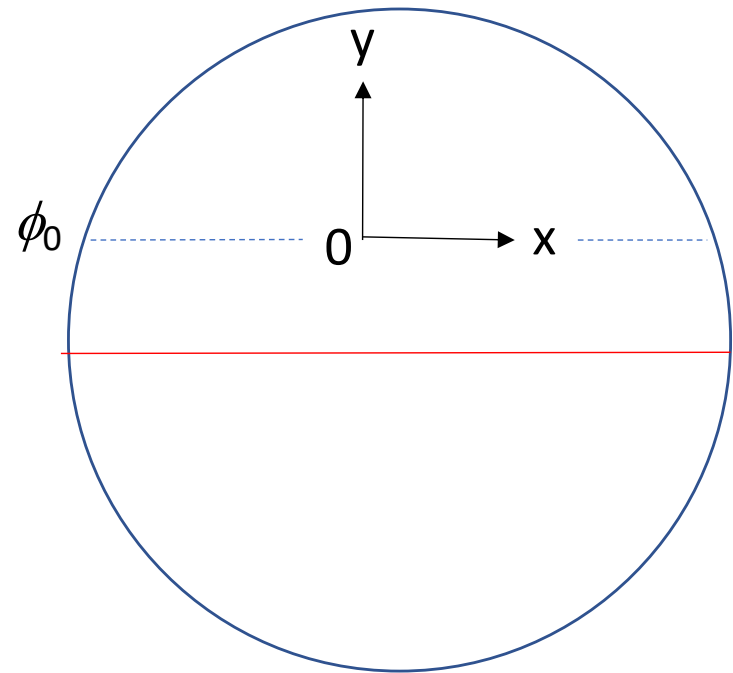
where

$$f_0 = 2\Omega \sin \phi_0$$

$$\beta = \frac{2\Omega}{a} \cos \phi_0 \sim \frac{df}{dy}$$

Mid-latitude beta-plane : $f = f_0 + \beta y$

Equatorial beta-plane : $f = \beta y$



Quasi-geostrophic vorticity equation

Introducing beta-plane approximation to the quasi-geostrophic Lagrangian derivative and retaining small quantities to the first order, the rate of change of the geostrophic wind is given by

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \vec{v}_g \cdot \nabla \right) \vec{v}_g &= -(f_0 + \beta y) \vec{k} \times (\vec{v}_g + \vec{v}_a) - \nabla \Phi \\ &\sim -f_0 \vec{k} \times \vec{v}_a - \beta y \vec{k} \times \vec{v}_g \end{aligned} \quad (1)$$

geostrophic flow
 $f_0 \vec{k} \times \vec{v}_g + \nabla \Phi = 0$

Since geostrophic wind is nondivergent ($\nabla \cdot \vec{v}_g = 0$), the continuity eq. is

$$\nabla \cdot \vec{v}_a + \frac{\partial \omega}{\partial p} = 0 \quad (2)$$

Operating rotation ($\nabla \times$) to (1) and using (2), we have the quasi-geostrophic vorticity equation.

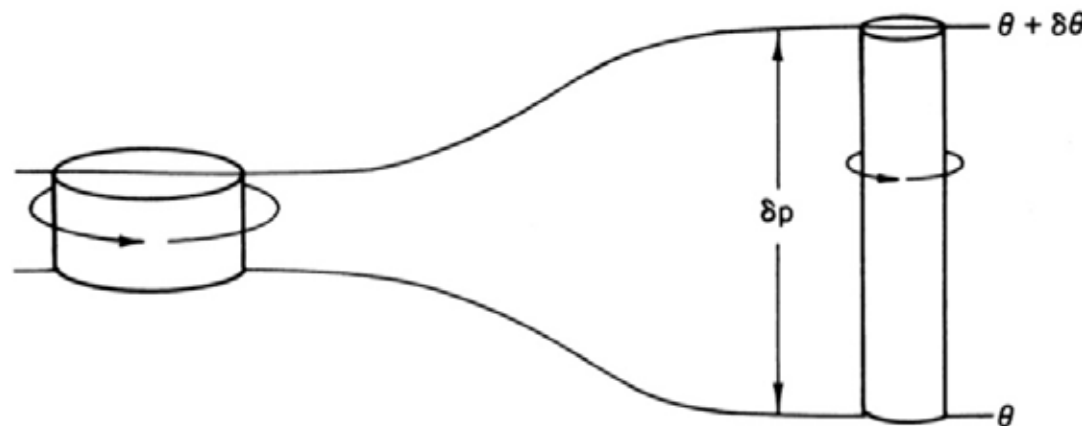
Quasi-geostrophic vorticity equation:

$$\frac{\partial \zeta_g}{\partial t} = -\vec{v}_g \cdot \nabla(\zeta_g + f) + f_0 \frac{\partial \omega}{\partial p}$$

$$\zeta_g \equiv \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{\nabla^2 \Phi'}{f_0} \quad : \text{geostrophic vorticity}$$

Vorticity changes with time through

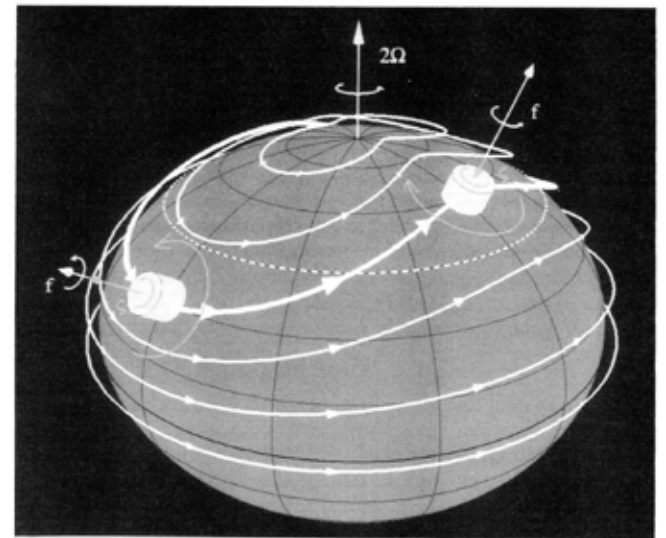
- advection of absolute vorticity ($\zeta_g + f$) by geostrophic wind ($\vec{v}_g$)
- vertical divergence (horizontal divergence)



Holton (2004)

Fig. 4.7 A cylindrical column of air moving adiabatically, conserving potential vorticity.

Rossby wave



Let us consider a two-dimensional motion ($\omega = 0$)

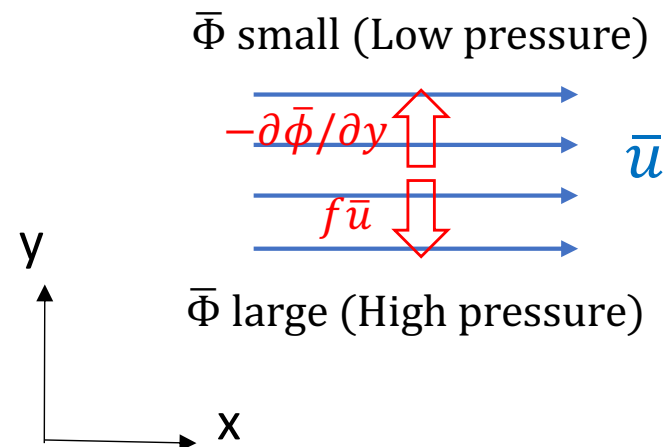
$$\left(\frac{\partial}{\partial t} + \vec{v}_g \cdot \nabla \right) (\zeta_g + f) = 0$$

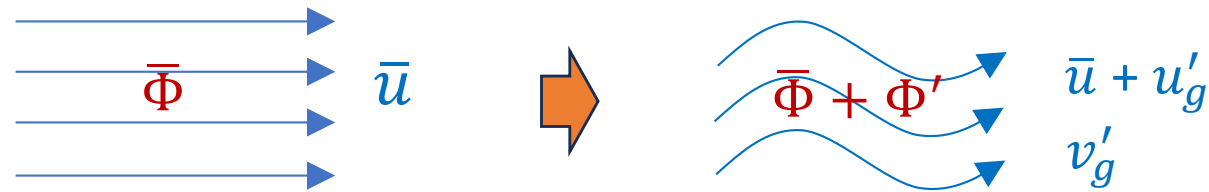
→ Absolute vorticity ($\zeta_g + f$) is conserved along the geostrophic wind $\vec{v}_g$.

A basic state where a homogeneous zonal flow exists:

$$f\bar{u} = -\partial\bar{\Phi}/\partial y$$

$$\bar{\zeta}_g = 0$$



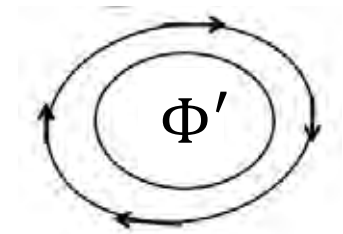


The deviation from the basic state is denoted by ()' :

$$\frac{\partial}{\partial t} \zeta_g + (\bar{u} + u'_g) \frac{\partial \zeta_g}{\partial x} + v'_g \frac{\partial \zeta_g}{\partial y} + v'_g \beta = 0 \quad (1)$$

The velocity and vorticity are related to the geopotential perturbation

$$\zeta_g = \frac{\nabla^2 \Phi'}{f_0}, \quad u'_g = -\frac{1}{f_0} \frac{\partial \Phi'}{\partial y}, \quad v'_g = \frac{1}{f_0} \frac{\partial \Phi'}{\partial x}$$



Substituting these into (1) and retaining first order terms only, we get

$$\frac{\partial}{\partial t} \nabla^2 \Phi' + \bar{u} \frac{\partial}{\partial x} \nabla^2 \Phi' + \beta \frac{\partial \Phi'}{\partial x} = 0$$

Assuming a wave solution $\Phi' = \hat{\Phi} \exp[i(kx + ly - \sigma t)]$, the phase velocity of **Rossby wave** is obtained as:

$$c - \bar{u} = -\frac{\beta}{k^2 + l^2}$$

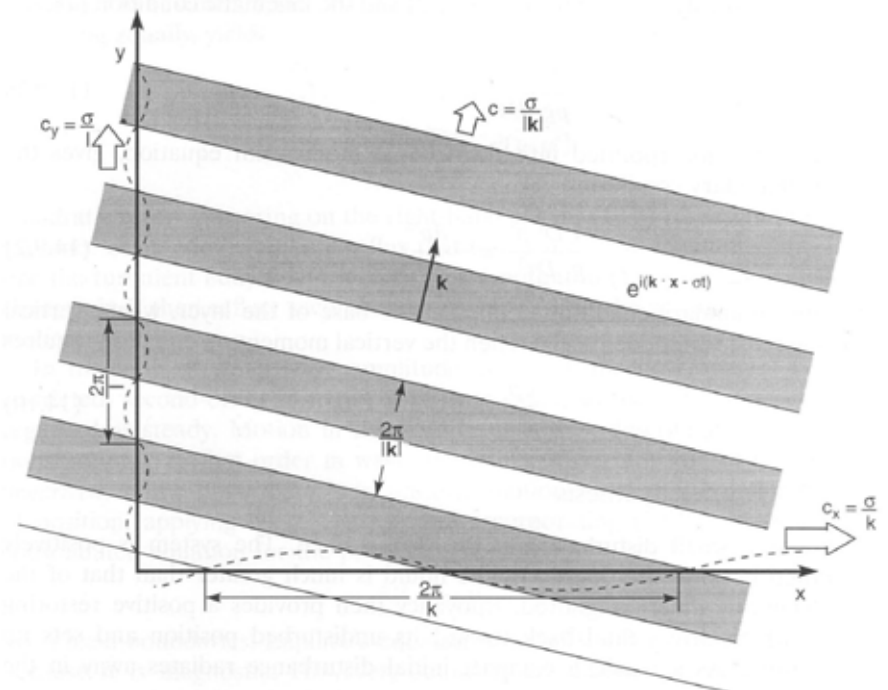
k : zonal wavenumber

l : meridional wavenumber

c : zonal phase velocity

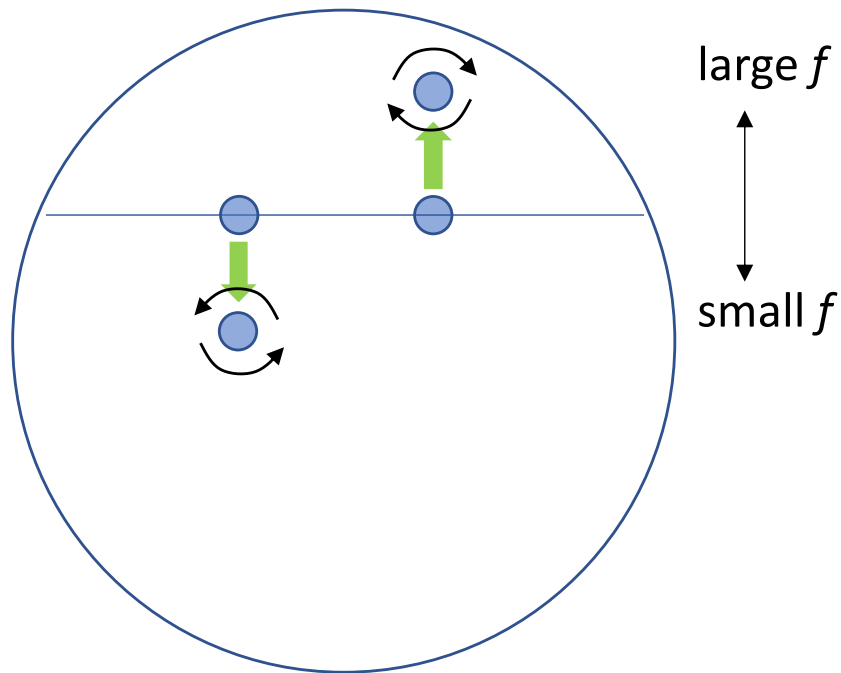
- Propagation opposite to the planetary rotation
- β effect (latitude variation of the Coriolis parameter f) is needed.
- Longer waves (smaller k) propagate faster.

The wave possesses angular momentum in the direction opposite to the planetary rotation



Salby (1996)

Propagation of Rossby wave



$$\left(\frac{\partial}{\partial t} + \vec{v}_g \cdot \nabla \right) (\xi_g + f) = 0$$

$$f = 2\Omega \sin \phi$$

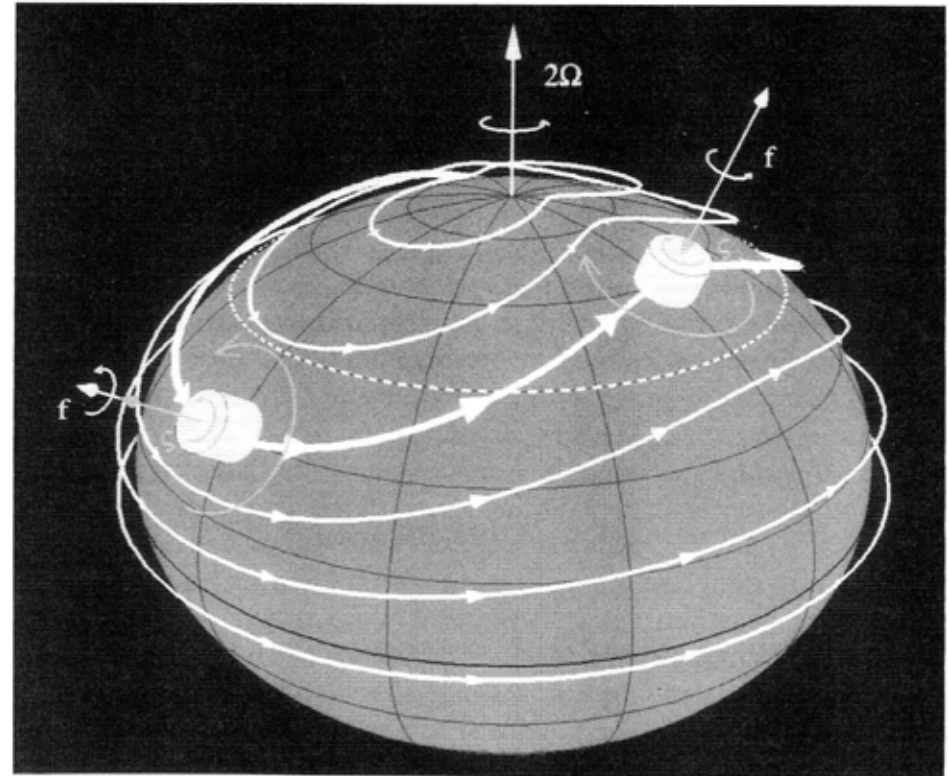
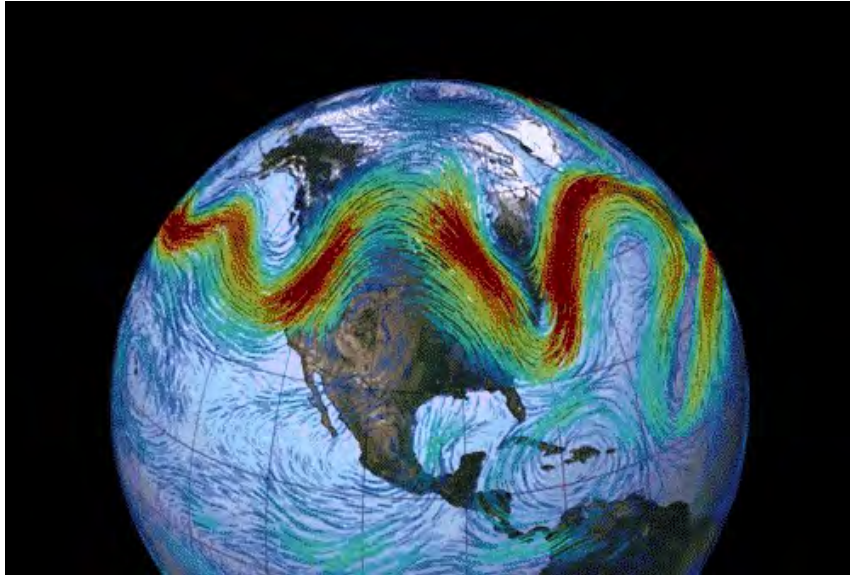


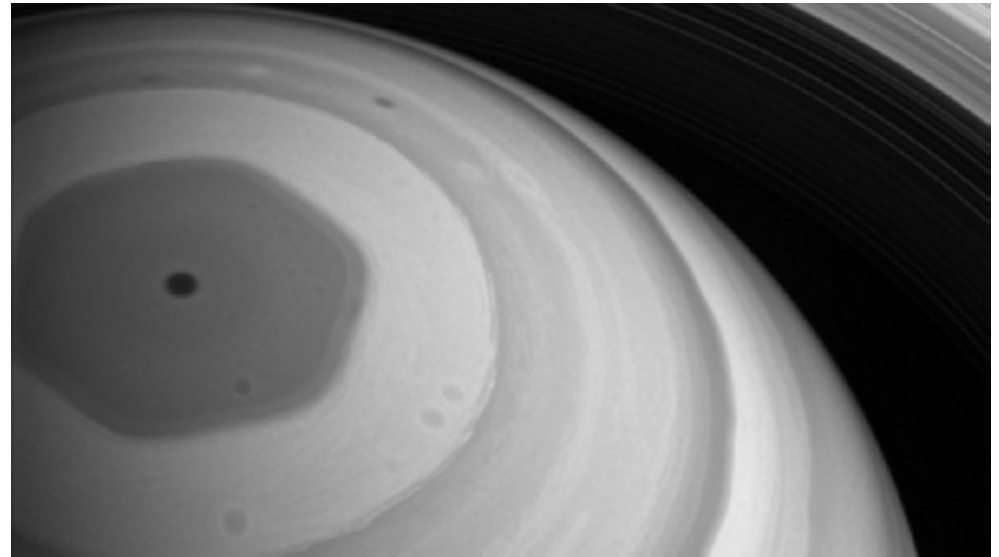
Figure 14.16 Schematic illustrating the reaction of an air parcel to meridional displacement. Displaced equatorward, an eastward-moving parcel spins up cyclonically to conserve absolute vorticity. Northward motion induced ahead of it then deflects the parcel's trajectory poleward back toward its undisturbed latitude. The reverse process occurs when the parcel overshoots and is displaced poleward of its undisturbed latitude.

Salby (1996)

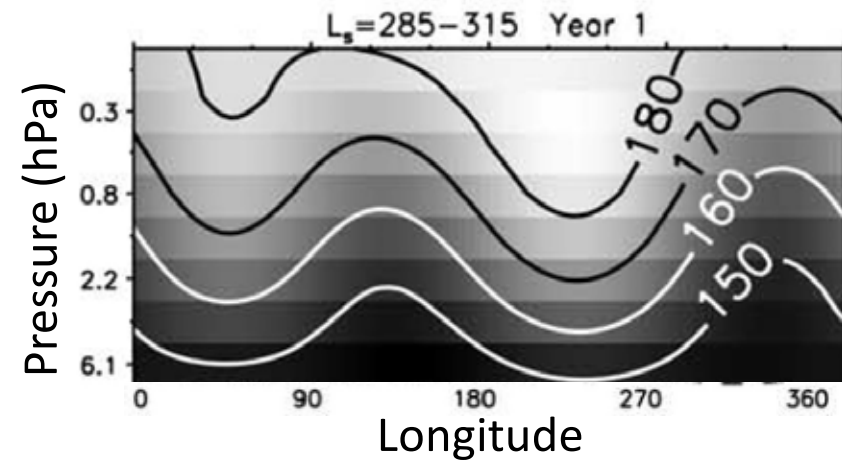
Rossby waves in planetary atmospheres



Earth

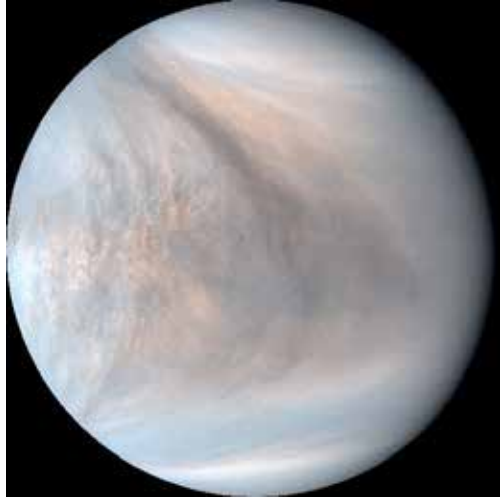


Saturn



Mars (MGS/TES temperature)

Rossby waves on Venus



Venus' 5-day wave observed by cloud-tracking
(Nara & Imamura 2025)

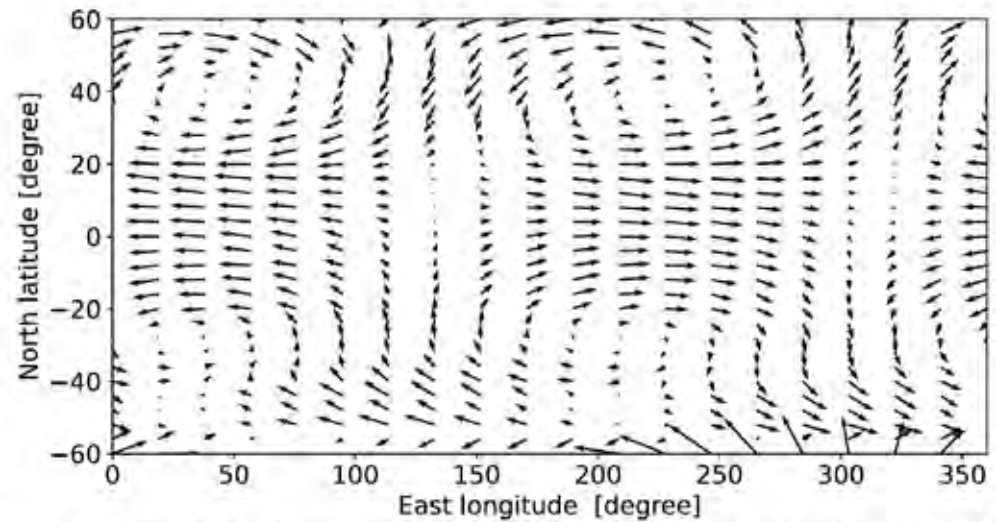
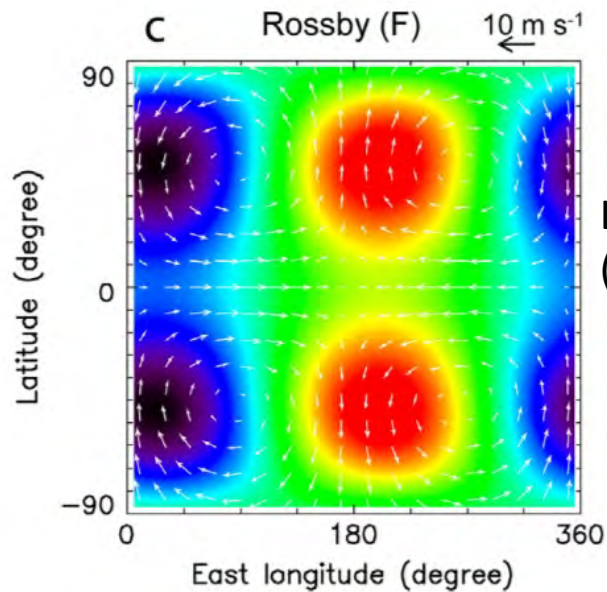


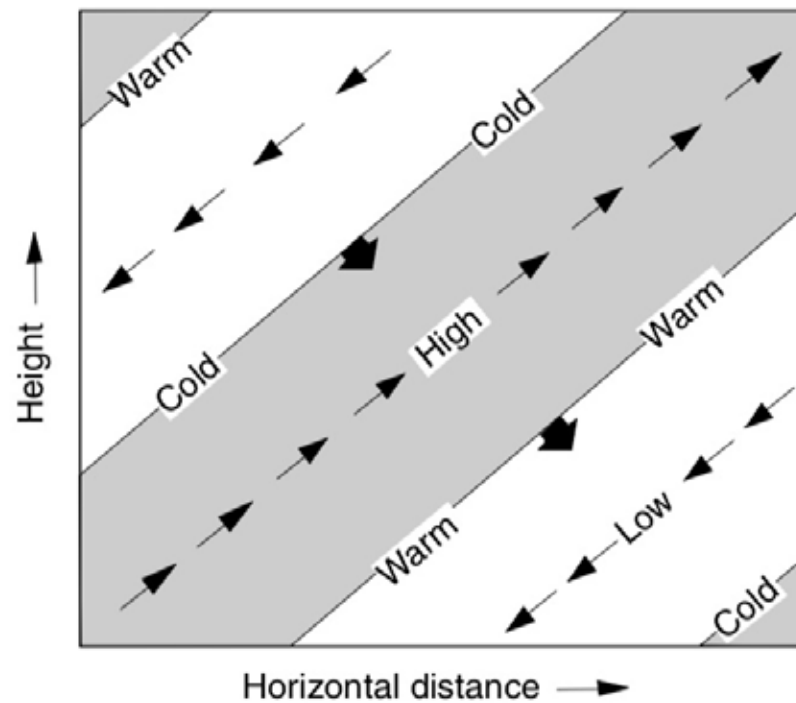
Fig. 9 Horizontal structure of the velocity perturbation with a period of 4.9 days. The arrows are scaled such that the distance occupied by 10° in latitude or longitude represents a speed of 5 m s^{-1} .



Linear solution of Rossby wave at 70 km
(Kouyama et al. 2015)

The superrotation of the atmosphere takes the place of planetary rotation.

Gravity wave



Holton 2004

Mars limb image taken by Mariner 9
(Anderson & Leovy 1978)

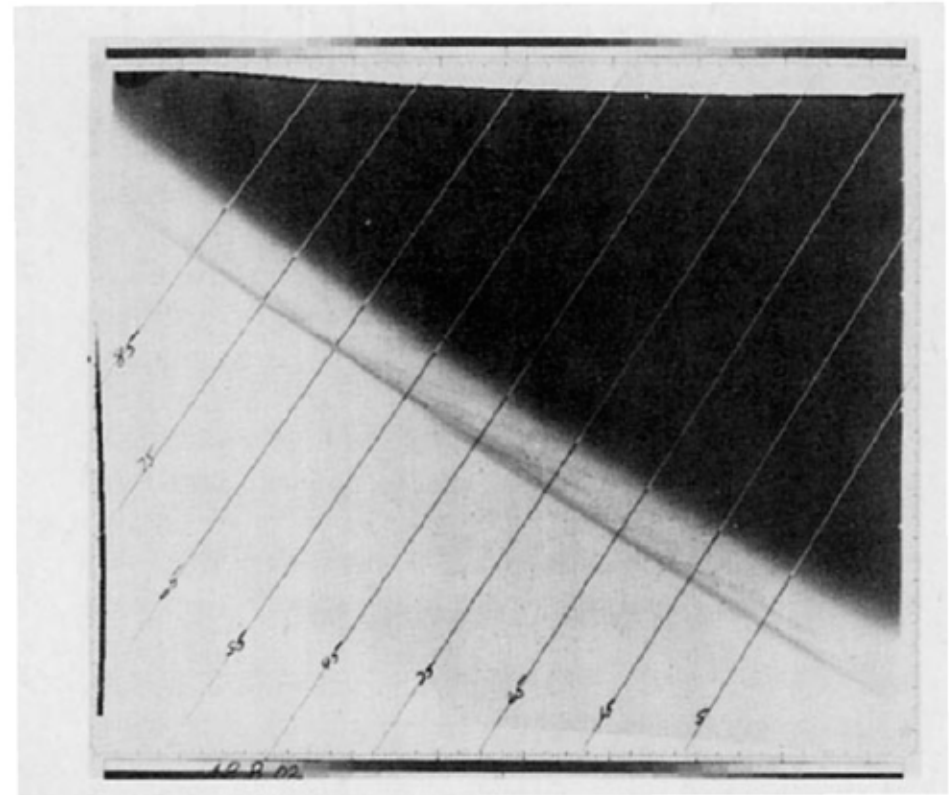


FIG. 10. Picture 48B02 (B-camera, rev 48) showing the limb in a negative version with planet at lower left, and haze layer running diagonally across the image. The limb is at 31°N , 336°W , and the local time is 1015.

Gravity wave

In Cartesian coordinates, without the assumption of hydrostatic equilibrium, the governing equations are:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial x}$$

horizontal momentum eq.
(x-axis only)

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z}$$

vertical momentum eq.

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho w)}{\partial z}$$

continuity eq.

$$\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + w \frac{\partial \theta}{\partial z} = 0$$

thermodynamics eq.

$$\theta = \frac{p}{\rho R} \left(\frac{p_s}{p} \right)^{R/C_p}$$

T

Equations for disturbances:

$$\bar{\rho} \frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x}$$

horizontal momentum eq.
(x-axis only)

$$\bar{\rho} \frac{\partial w'}{\partial t} = -\frac{\partial p'}{\partial z} - \rho' g$$

vertical momentum eq.

$$\frac{\partial \rho'}{\partial t} = -\bar{\rho} \frac{\partial u'}{\partial x} - \frac{\partial (\bar{\rho} w')}{\partial z}$$

continuity eq.

$$\frac{1}{\bar{\theta}} \frac{\partial \theta'}{\partial t} + w' \frac{N^2}{g} = 0$$

thermodynamics eq.

$$\rho' = \frac{p'}{c_s^2} - \bar{\rho} \frac{\theta'}{\bar{\theta}}$$

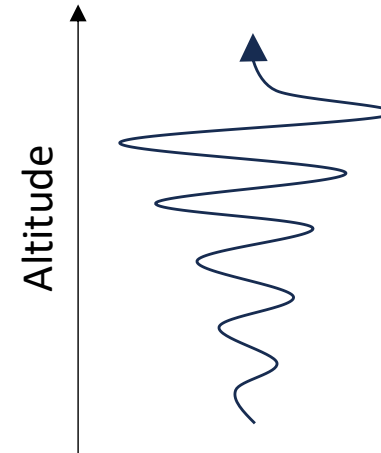
$$c_s^2 = \frac{c_p}{c_v} RT \quad c_s: \text{sound speed}$$

$$N^2 = g \frac{d \ln \bar{\theta}}{dz} \quad N: \text{buoyancy frequency}$$

Assuming an isothermal atmosphere:

$$N^2 = g/c_p T$$

$$\bar{\rho}(z) = \rho_s \exp(-z/H)$$



Substituting the wave solution $w'(x, z, t) = \hat{w}(z) \exp[i(kx + \sigma t)]$ (σ : frequency) into the governing equations before, an equation for the vertical velocity w is obtained as

$$\frac{d^2(\bar{\rho}\hat{w})}{dz^2} + \frac{1}{H} \frac{d(\bar{\rho}\hat{w})}{dz} + \left[\frac{\sigma^2}{c_s^2} - k^2 + \frac{N^2 k^2}{\sigma^2} \right] \bar{\rho}\hat{w} = 0$$

Considering the amplitude growth with height in a stratified atmosphere, w is assumed to have the form

$$\hat{w}(z) = W(z) \exp(z/2H) \quad \Leftrightarrow \quad \bar{\rho}\hat{w}^2(z) \propto W^2(z)$$

Then the equation becomes

$$\frac{d^2W}{dz^2} + \left[\frac{\sigma^2}{c_s^2} - k^2 + \frac{N^2k^2}{\sigma^2} - \frac{1}{4H^2} \right] W = 0$$

Assuming a wave solution $W \propto \exp(imz)$ (m : vertical wavenumber), the **dispersion relation** is obtained:

$$m^2 = \frac{\sigma^2}{c_s^2} - k^2 + \frac{N^2k^2}{\sigma^2} - \frac{1}{4H^2}$$

Solutions for acoustic-gravity wave and internal gravity wave exist.

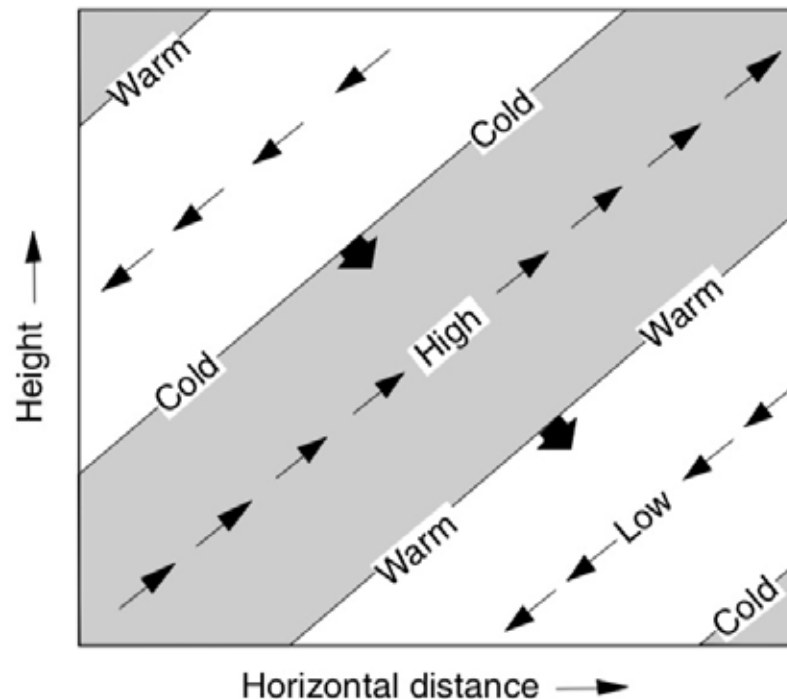
Approximate solution for internal gravity wave is

$$\sigma^2 = \frac{N^2k^2}{k^2 + m^2 + \frac{1}{4H^2}} \quad \rightarrow \quad \sigma < N$$

Structure of gravity wave

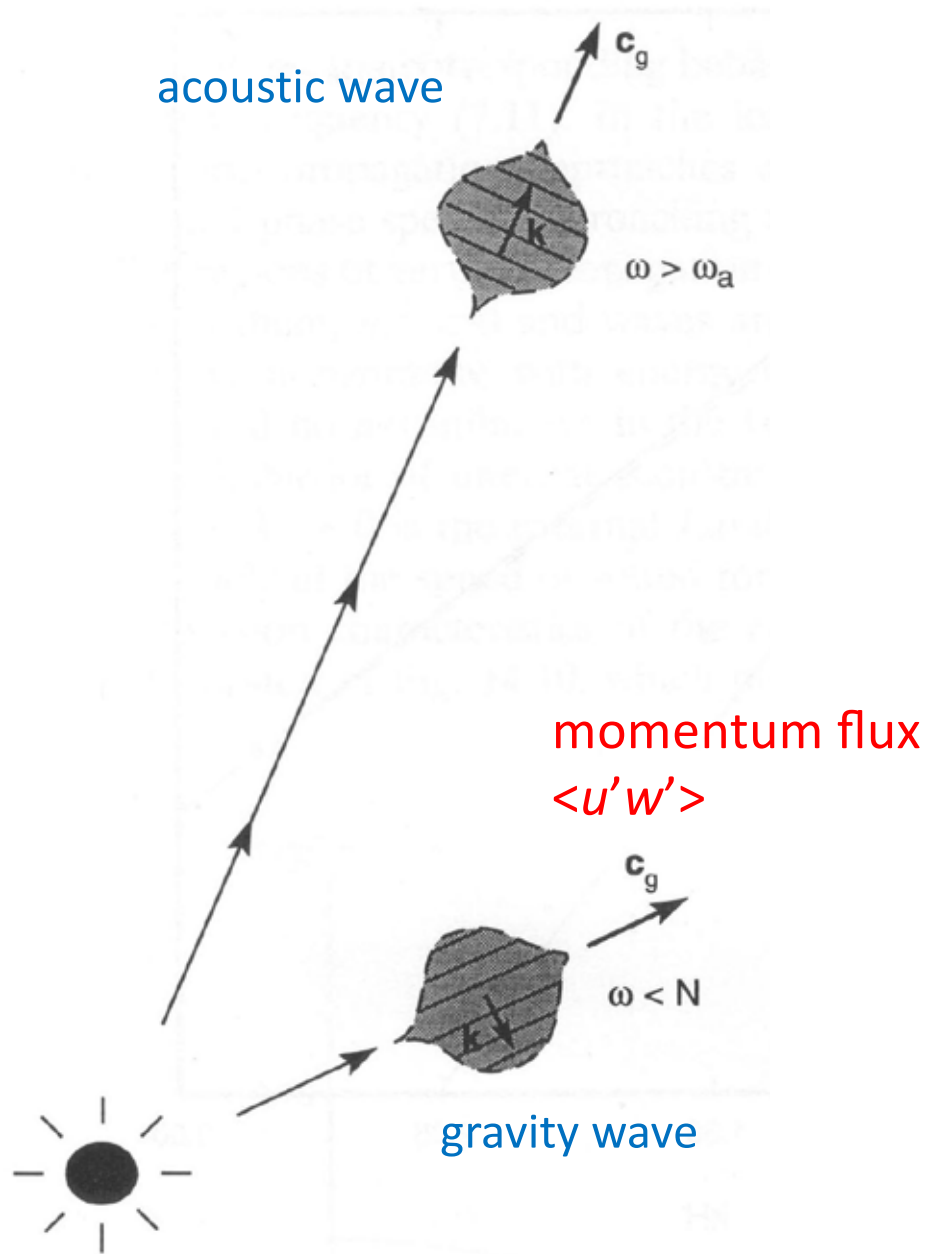
For large-horizontal scale waves (typical in planetary atmospheres),

$$\left| \frac{k}{m} \right| = \left| \frac{\sigma}{N} \right| \quad \rightarrow \text{Long period waves have near-horizontal phase surfaces}$$

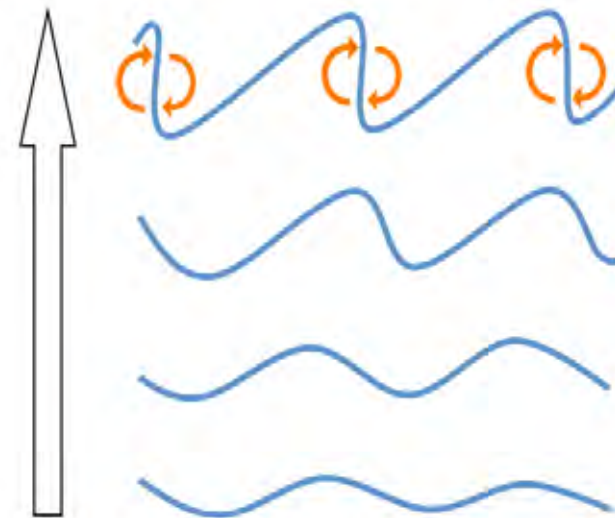


Holton 2004

Fig. 7.9 Idealized cross section showing phases of pressure, temperature, and velocity perturbations for an internal gravity wave. Thin arrows indicate the perturbation velocity field, blunt solid arrows the phase velocity. Shading shows regions of upward motion.

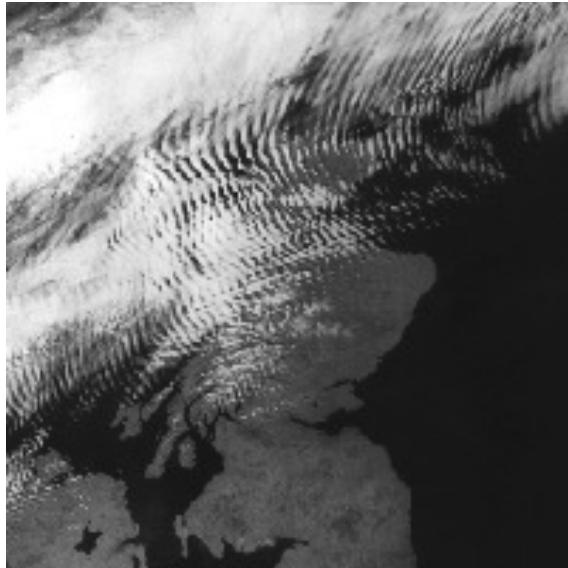


Salby (1996)

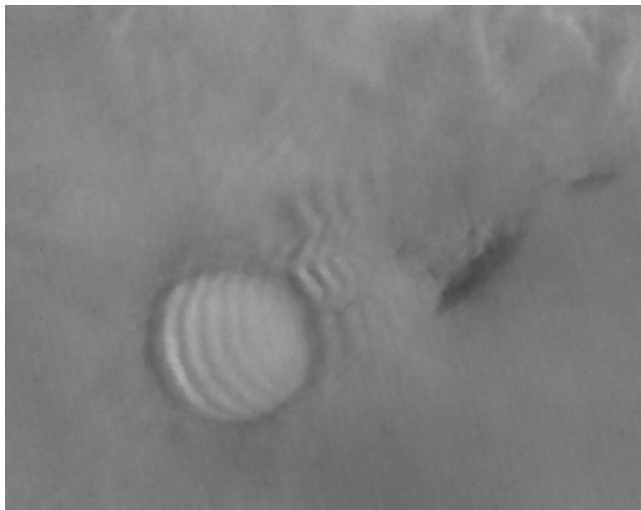


Amplitude growth with height
and wave breaking
→ turbulence generation,
mean-wind acceleration

Observed gravity waves



Mountain waves on Earth



Mountain waves on Mars

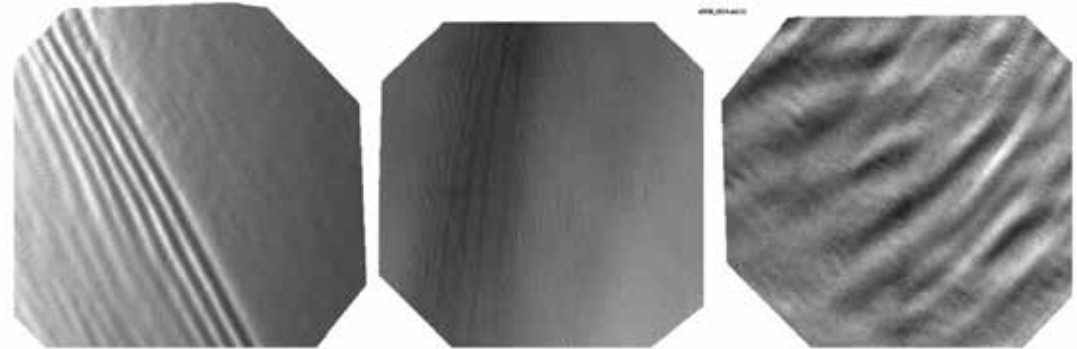
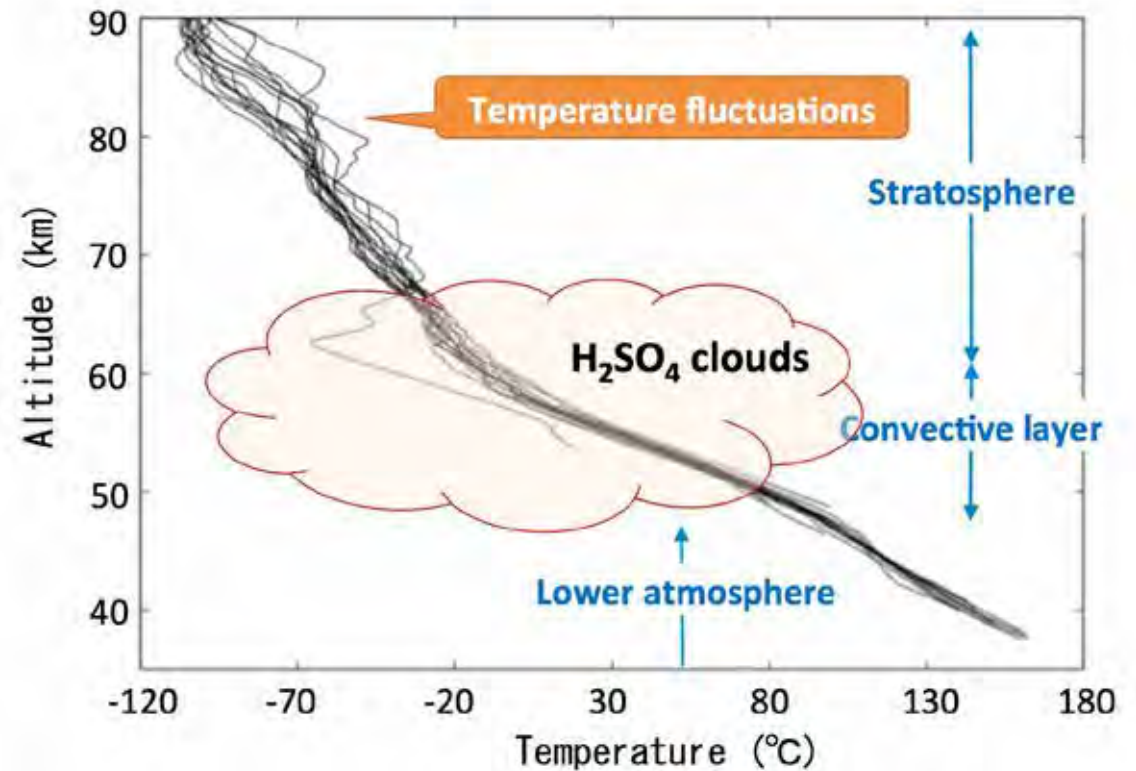
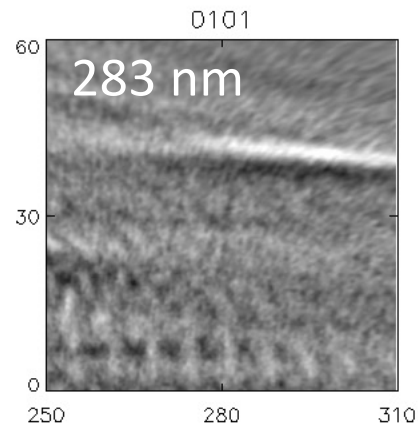


Fig. 11. VMC images of polar waves: left – long waves (NIR1 filter), middle – long waves producing short wave trains (UV), right – irregular waves (UV).

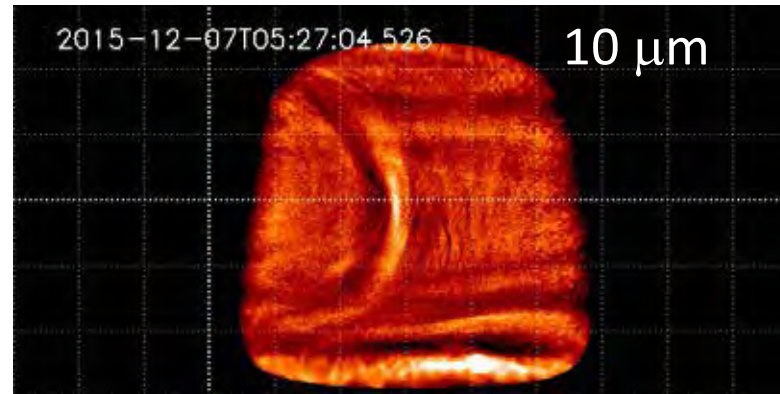
on Venus



Topographically-generated gravity waves (mountain waves) on Venus

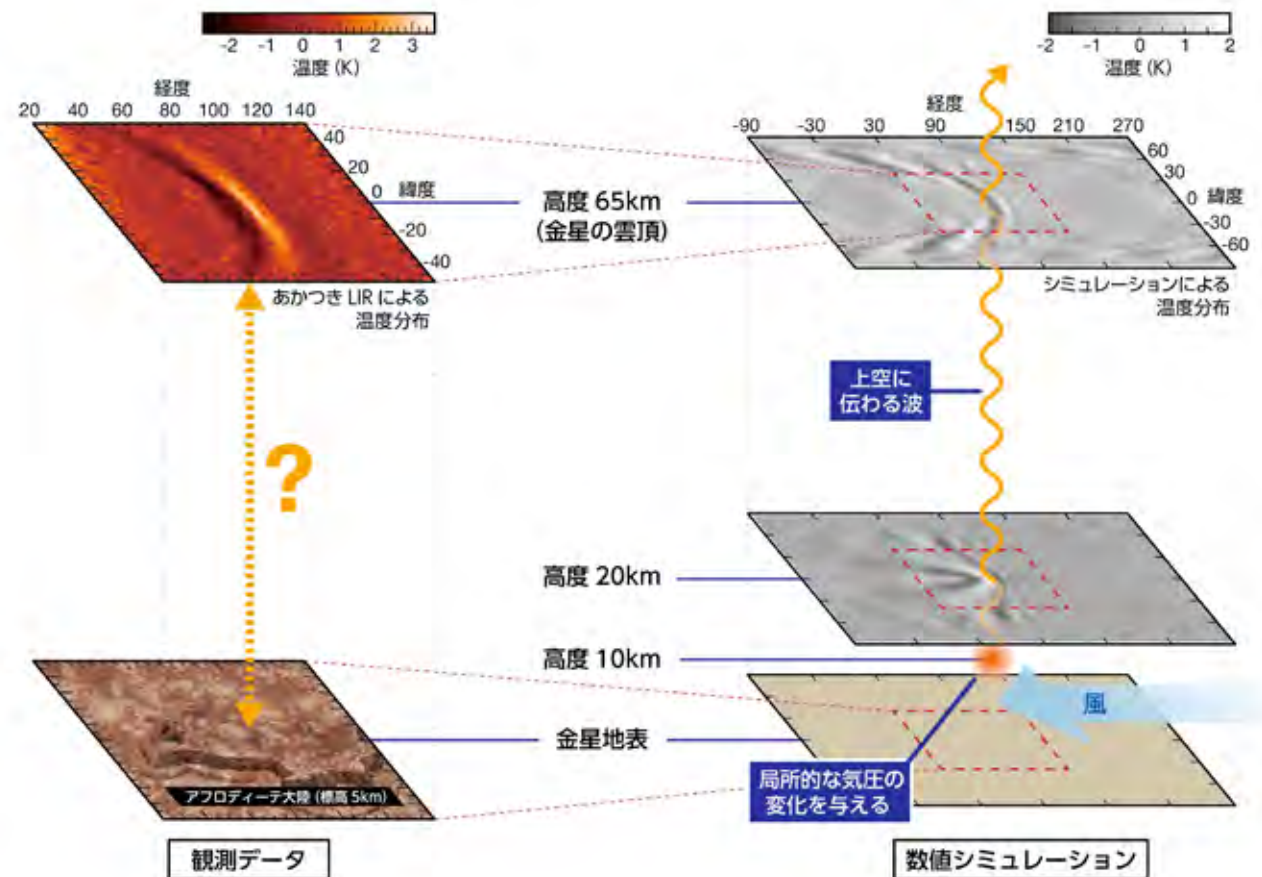


Kitahara et al. (2020)



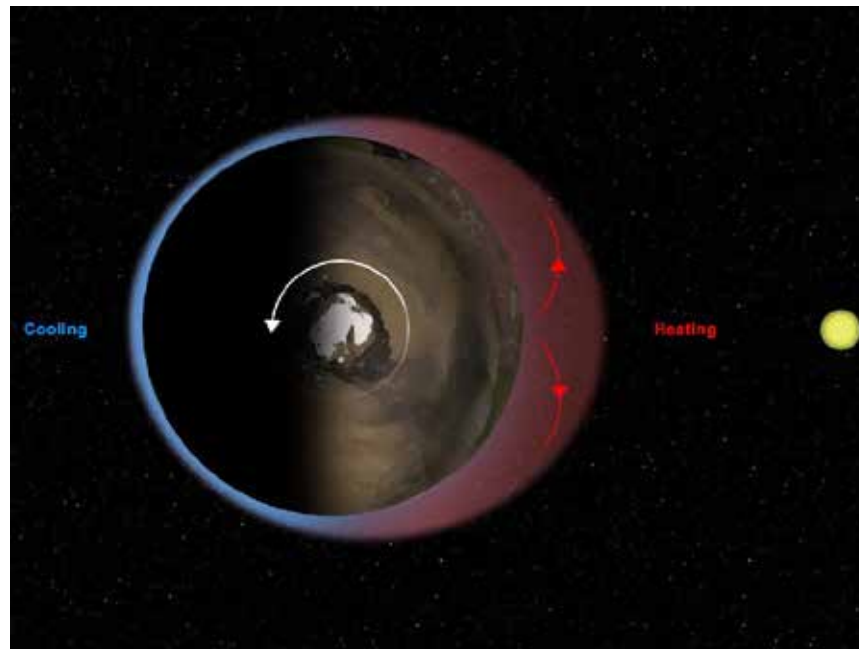
Fukuhara et al. 2017

Kouyama et al. 2017



Thermal tides

Planetary-scale gravity wave generated by the movement of the solar heating region in the diurnal cycle



©NASA

Excitation mechanism:

- Earth : solar heating of stratospheric ozone layer
- Venus : solar heating of cloud layer
- Mars : solar heating of atmospheric dust

Thermal tide in Earth's atmosphere

Vertical structure

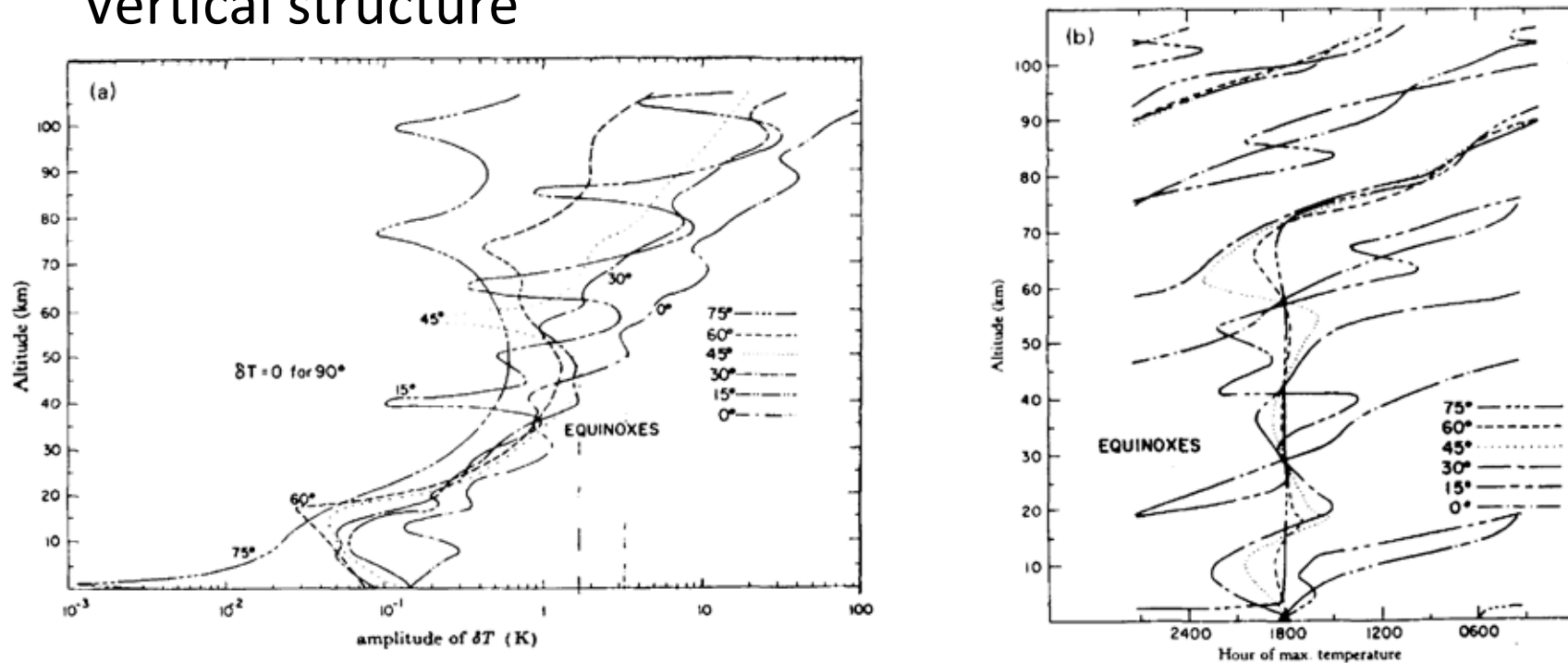
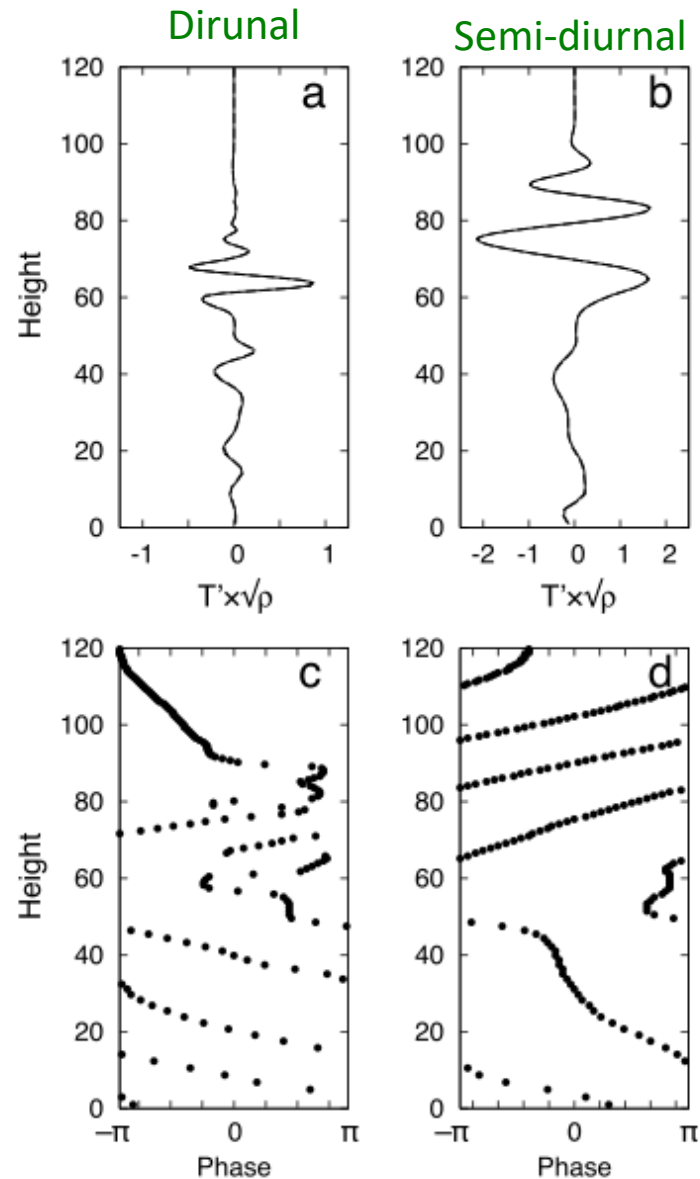


Fig. 4.7. (a) Amplitude and (b) phase of solar diurnal component of T' at various latitudes for equinox. [After Lindzen (1967).]

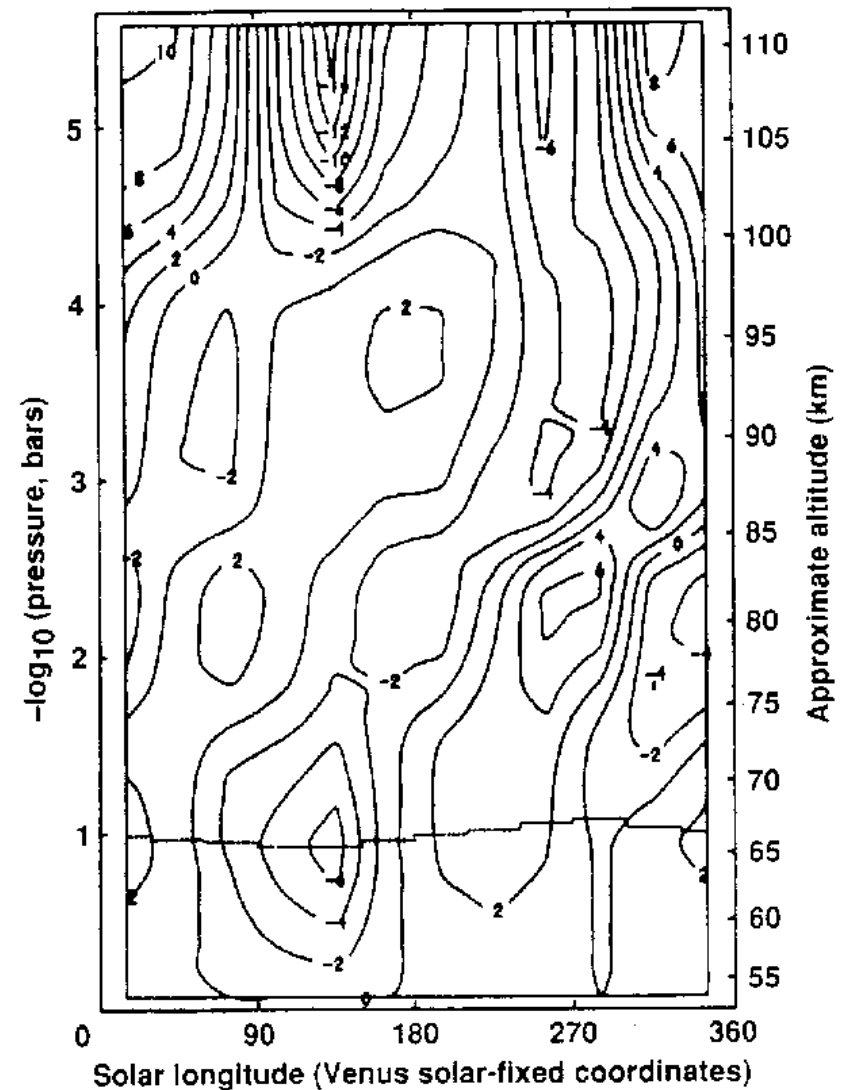
Holton (1992)

Thermal tide in Venus' atmosphere

Linear solution (Takagi & Matsuda, 2006)

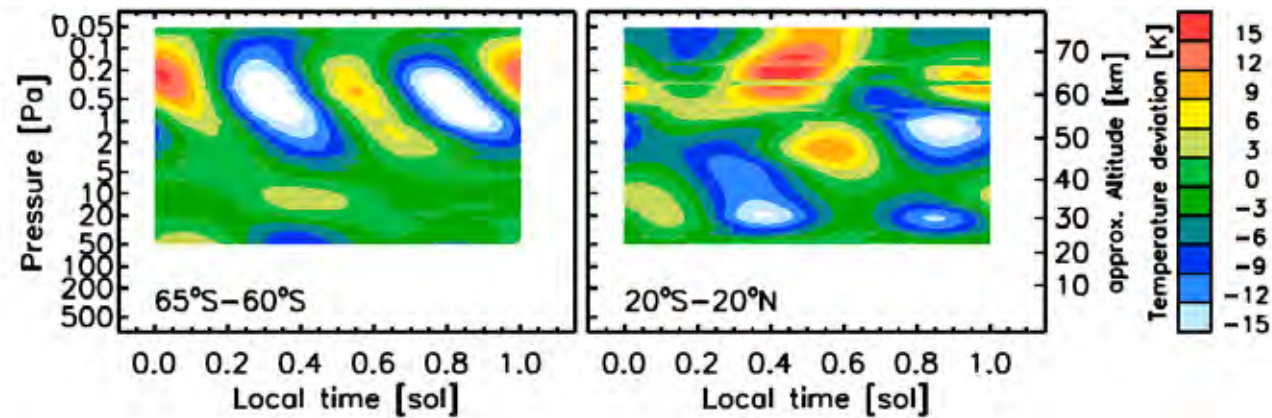


Temperature perturbation
(Schofield & Taylor 1983)



Thermal tide in Mars' atmosphere

Wavenumber 1+2
components observed
by MCS onboard MRO



Mars GCM

